5. A little topology

We need to say a few words about topology. We will be interested in functions that are not defined on the whole complex plane. Since we want to differentiate, we want to take limits. On the one hand, derivatives are purely local about a point z_0 but on the other hand to take a limit you want to approach from all sides. We try to capture the right domains for our functions in the following definitions.

Definition 5.1. *Let z_0 be a complex number and let $\epsilon > 0$ be a positive real.*

The ***open disk*** *centred about z_0 of radius ϵ is the set*

$$\{ z \in \mathbb{C} \,|\, |z - z_0| < \epsilon \}.$$

Note that if you take any point z_1 of an open disk it is clear that one can approach z_1 in any direction so that limits make sense on an open disk. Note also that one can take ϵ as small as you please, which captures the local nature of taking limits.

An open set is then just a union of open disks:

Definition 5.2. *Let $U \subset \mathbb{C}$ be a subset.*

We say that U is ***open*** *if it is a union of open disks.*

We allow the empty union so that the empty set is open.

There is a companion notion to open:

Definition 5.3. *Let $F \subset \mathbb{C}$ be a subset.*

We say that F is ***closed*** *if the complement $\mathbb{C} \setminus U$ is open.*

Definition-Lemma 5.4. *Let z_0 be a complex number and let $\epsilon > 0$ be a positive real.*

The ***closed disk*** *centred about z_0 of radius ϵ is the set*

$$\{ z \in \mathbb{C} \,|\, |z - z_0| \leq \epsilon \}.$$

A subset $F \subset \mathbb{C}$ is closed if and only if it is the intersection of closed disks.

We allow the empty intersection so that $F = \mathbb{C}$ is closed.

It is also useful to think of bounded sets

Definition 5.5. *Let $X \subset \mathbb{C}$ be a subset.*

We say that X is ***bounded*** *if it is contained in a disk.*

In fact X is bounded if and only if it is contained in a disk centred the origin.

We would like to require one more property of the domains. It is useful to require connectedness.

Definition 5.6. *We say that $X \subset \mathbb{C}$ is* ***connected*** *if every continuous function*

$$f\colon X \longrightarrow \{0, 1\}$$

is constant.

Here $\{0, 1\}$ is a set with two points 0 and 1 sitting inside the real numbers (one can also say that f is function from X to the real number whose image is containe in the set $\{0, 1\}$). Roughly speaking a set X is disconnected if you can break it into two completely separate pieces.

Definition 5.7. *A* ***region*** *$U \subset \mathbb{C}$ is an open connected subset of the complex plane.*

Most of our functions will be defined on a region. There are lots of examples of regions. The open disk, the upper half plane, vertical strips, horizontal strips, angular regions, the complement of a closed disk, and so on.

Note that the comolement of a line is open but not a region (it has two connected components). The complement of a half line or a closed bounded line segment is a region.

It is also worth noting the following:

Definition 5.8. *We say that $X \subset \mathbb{C}$ is* ***path connected*** *if for any two points x_0 and x_1 there is a* ***path****, that is, a continuous function*

$$\gamma\colon [0, 1] \longrightarrow X,$$

and $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

It is much easier to check if a set X is path connected, rather than connected. It is also not hard to prove that path connected implies connected. However the converse is not true in general, there are connected sets which are not path connected.

Example 5.9. *Let*

$$X = \{\, (t, \sin 1/t) \,|\, t \in (0, \infty) \,\} \cup \{\, (0, y) \,|\, y \in (-\infty, \infty) \,\},$$

that is, the union of the graph of

$$\mathbb{R} \longrightarrow \mathbb{R} \qquad \text{given by} \qquad t \longrightarrow \sin 1/t$$

and the y-axis. Then X is not path connected. It is impossible to get from a point of the graph to a point of the y-axis. The problem is that $\sin 1/t$ oscillates more and more as t gets small. On the other hand, it is not hard to check that X is connected.

However, we do have

Proposition 5.10. *Let $U \subset \mathbb{U}$ be open.*

Then U is connected if and only if it is path connected.

Let us end with one of the jewels of topology.

Definition 5.11. *A closed path if a continuous function*

$$\gamma \colon [0,1] \longrightarrow \mathbb{C},$$

such that $\gamma(0) = \gamma(1)$.

We say that γ is simple if $\gamma(t_0) = \gamma(t_1)$ implies that either $t_0 = t_1$ or that t_0 and t_1 are endpoints of $[0,1]$.

Closed just means that you come back to where you started and simple means that you only do this at the endpoints, the path does not cross itself.

Theorem 5.12 (Jordan curve theorem)**.** *A simple closed curve separates $\mathbb{C}$ into two regions.*

Separate means that we consider the complement of the image of the simple closed path. The regions are called the inside and the outside. The inside is bounded. Despite the fact that (5.12) looks obvious it is suprisingly hard to prove.