4. Roots of unity

Theorem 4.1 (De Moivre's Theorem)**.**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Proof. We have

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (e^{i\theta})^n \\ &= e^{in\theta} \\ &= \cos n\theta + i\sin n\theta.\end{aligned}$$ □

One can use this to derive simple formulas. For example suppose we want to compute triple angle formulas. We use (4.1) to when $n = 3$. We can expand the LHS using the binomial theorem.

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

Equating real and imaginary parts we get

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta.\end{aligned}$$

We can also use Euler's formula to compute nth roots.

Example 4.2. *What are the cube roots of* 125*?*

We are looking for complex numbers z such that

$$z^3 = 125.$$

We write z in polar form

$$z = re^{i\theta}.$$

Then we get the equation

$$r^3 e^{3i\theta} = 125.$$

Taking the modulus of both sides we see that

$$r^3 = 125.$$

As r is a non-negative real number it follows that

$$r = 5.$$

If we cancel 125 from both sides, we are reduced to solving

$$e^{3i\theta} = 1,$$

that is, we are trying to find all cube roots of 1.

What are the possible arguments of such complex numbers? One possibility is clear, $\theta = 0$. In other words, 1 is a cube root of one. Another possibility is that

$$3\theta = 2\pi,$$

so that when we add θ to itself we go once around the origin. This gives the solution

$$\theta = \frac{2\pi}{3}.$$

It follows that

$$\omega = \frac{1}{2}\left(-1 + \sqrt{3}i\right)$$

is a cube root of one.

A third possibility is that we go twice around the origin, so that

$$3\theta = 4\pi \qquad \text{and} \qquad \theta = \frac{4\pi}{3}.$$

In this case we get the last cube root of one

$$\omega' = \frac{1}{2}\left(-1 - i\sqrt{3}\right).$$

Note some interesting connections between the roots. First off ω' is the complex conjugate of ω:

$$\omega' = \bar{\omega}.$$

In fact it is a general fact that the roots of a real polynomial come in complex conjugate pairs.

Lemma 4.3. *Let $p(x)$ be a real polynomial.*

Then the roots of $p(x)$ come in complex conjugate pairs.

Proof. We may suppose that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_0, a_1, \dots, a_n$ are real numbers. Let α be a root of $p(x)$. We have

$$\begin{aligned} p(\bar{\alpha}) &= a_n(\bar{\alpha})^n + a_{n-1}(\bar{\alpha})^{n-1} + \cdots + a_0 \\ &= a_n\overline{\alpha^n} + a_{n-1}\overline{\alpha^{n-1}} + \cdots + a_0 \\ &= \overline{a_n\alpha^n} + \overline{a_{n-1}\alpha^{n-1}} + \cdots + \overline{a_0} \\ &= \overline{a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0} \\ &= \overline{p(\alpha)} \\ &= \overline{0} \\ &= 0. \end{aligned}$$

Thus $\bar{\alpha}$ is also a root of $p(x)$. □

For example, for the polynomial

$$z^3 - 1 = (z-1)(z-\omega)(z-\omega'),$$

if we take the complex conjugate, the LHS is unchanged. On the RHS, 1 is fixed and so complex conjugation must switch ω and ω'.

Secondly note that

$$\omega' = \omega^2.$$

This is again a general fact:

Lemma 4.4. *If ζ is an nth root of unity then so are all powers of ζ.*

Proof. Consider $\alpha = \zeta^a$, where a is a non-negative integer. We have

$$\begin{aligned} \alpha^n &= (\zeta^a)^n \\ &= \zeta^{an} \\ &= (\zeta^n)^a \\ &= 1^a \\ &= 1. \end{aligned}$$

□

Note that there is a simple relation between ω and $\omega' = \omega^2$. Playing around a little bit one sees that

$$-\omega^2 = 1 + \omega,$$

so that

$$\omega^2 + \omega + 1 = 0.$$

In fact

$$z^3 - 1 = (z-1)(z^2 + z + 1),$$

as can be seen from direct calculation.

There are similar pictures for 4th and 5th roots. The 4th roots are ± 1 and $\pm i$. i and $-i$ are complex conjugates.

$$i = e^{i\pi/2}$$

and the other roots are powers of i:

$$i = i^1 \qquad -1 = i^2 \qquad -i = i^3 \qquad \text{and} \qquad 1 = i^4.$$

± 1 are the square roots of 1. In fact we have

$$\begin{aligned} z^4 - 1 &= (z^2 - 1)(z^2 + 1) \\ &= (z-1)(z+1)(z^2+1). \end{aligned}$$

$\pm i$ are roots of $z^2 + 1$.

The fifth roots of 1 are

$$e^{i2m\pi/5},$$

where $m = 0$, 1, 2, 3 and 4. These are all powers of

$$\zeta = e^{i2\pi/5}.$$

ζ and ζ^4 are complex conjugates and so are ζ^2 and ζ^3. We have

$$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + 1),$$

so that ζ is a root of

$$z^4 + z^3 + z^2 + 1.$$