3. Complex exponentials

The exponential function from real variable has a power series expansion

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots.$$

Let us make a leap of faith and simply replace the real number x by the complex number z.

Definition 3.1. *The **exponential function** is the function*

$$\mathbb{C} \longrightarrow \mathbb{C} \qquad \textit{given by} \qquad z \longrightarrow e^z,$$

where

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots.$$

For the time being we will ignore all issues of convergence. It is clear how to define the sine and cosine

Definition 3.2. *The **sine function** is the function*

$$\mathbb{C} \longrightarrow \mathbb{C} \qquad \textit{given by} \qquad z \longrightarrow \sin z,$$

where

$$\sin z = z - \frac{z^3}{3!} + \dots.$$

Similarly

Definition 3.3. *The **cosine function** is the function*

$$\mathbb{C} \longrightarrow \mathbb{C} \qquad \textit{given by} \qquad z \longrightarrow \cos z,$$

where

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots.$$

These three functions clearly extend the usual three functions. Suppose we substitute iz for z;

$$\begin{aligned}
e^{iz} &= 1 + iz + \frac{(iz)^2}{2} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots + \frac{(iz)^n}{n!} + \dots \\
&= 1 + iz + \frac{-z^2}{2} + \frac{-iz^3}{3!} + \frac{z^4}{4!} + \dots + \frac{i^n z^n}{n!} + \dots \\
&= 1 + \frac{-z^2}{2} + \frac{z^4}{4!} + \dots + iz + \frac{-iz^3}{3!} + \frac{iz^5}{5!} + \dots \\
&= 1 - \frac{z^2}{2} + \frac{z^4}{4!} + \dots + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots) \\
&= \cos z + i \sin z.
\end{aligned}$$

Now suppose we replace z by the real number θ. Then we get a very famous formula due to Euler:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This makes using polar coordinates very easy:

$$\begin{aligned} z &= r\cos\theta + ri\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta}. \end{aligned}$$

It is fun to plug in simple values of θ, with $r = 1$, to see what we get. If we plug in $\theta = 0$ we get

$$\begin{aligned} 1 &= \cos 0 + i\sin 0 \\ &= e^{i0}; \end{aligned}$$

for $\theta = \pi/2$ we get

$$\begin{aligned} i &= \cos\pi/2 + i\sin\pi/2 \\ &= e^{i\pi/2}; \end{aligned}$$

$\theta = \pi/4$ we get

$$\begin{aligned} \frac{1}{\sqrt{2}}(1+i) &= \cos\pi/4 + i\sin\pi/4 \\ &= e^{i\pi/4}; \end{aligned}$$

$\theta = \pi/3$ we get

$$\begin{aligned} \frac{1}{2}\left(1+\sqrt{3}\right) &= \cos\pi/3 + i\sin\pi/3 \\ &= e^{i\pi/3}, \end{aligned}$$

and so on.

Perhaps the most interesting value to try is $\theta = \pi$. We get

$$\begin{aligned} -1 &= \cos\pi + i\sin\pi \\ &= e^{i\pi}. \end{aligned}$$

Rearranging gives one of the most beautiful formulas in all of mathematics

$$e^{i\pi} + 1 = 0.$$

The five fundamental constants of mathematics

$$1 \qquad 0 \qquad \pi \qquad e \qquad \text{and} \qquad i,$$

connected by one single equation.

The most convenient aspect of this way to represent complex number is that it is easy to multiply and at the same time it gives geometric meaning to multiplication. The key identity is what happens when you multiply two complex numbers of modulus one:

$$e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}.$$

In words just add the angles.

Assuming this formula for the time being, we get a simple way to multiply two complex numbers

$$z_1 = r_1 e^{i\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{i\theta_2}.$$

We have

$$\begin{aligned} z_1 z_2 &= (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) \\ &= r_1 r_2 e^{i\theta_1} e^{i\theta_2} \\ &= r_1 r_2 e^{i\theta_1+\theta_2}. \end{aligned}$$

In words multiply the modulus (the usual way) and add the argument.

Since

$$i = e^{i\pi/2}$$

multiplication by i represents rotation through $\pi/2$. More generally, multiplication by a complex number of modulus one represents rotation about the origin through the argument.

We now turn to a justification of the formula. We start with a mundane proof and then to indicate how to give a more interesting proof. For the straightforward proof, we just use the addition formula:

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta. \end{aligned}$$

We have

$$\begin{aligned} e^{i\alpha} \cdot e^{i\beta} &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta) \\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\ &= e^{i(\alpha+\beta)}. \end{aligned}$$

To illustrate how to give a different proof of the above formula, we give two different ways to prove the identity:

$$\cos^2 z + \sin^2 z = 1.$$

The first method uses the fact that this identity is well-known if z is a real number.

Both sides expand to power series in z and we just have to show that these power series

$$\sum a_n z^n \qquad \text{and} \qquad \sum b_n z^n = 1$$

are equal. Now the power series on the RHS is the trivial one,

$$b_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

So we just need to show the power series on the LHS is the same trivial power series. If we substitute in the value $z = x$ then we do get equality

$$\cos^2 x + \sin^2 x = 1.$$

This gives an equality of power series

$$\sum a_n x^n = 1.$$

But then

$$a_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0. \end{cases}$$

It follows that

$$\sum a_n z^n = 1.$$

But then

$$\cos^2 z + \sin^2 z = 1.$$

The key point is that a power series

$$\sum a_n z^n$$

is determined by its values on the real line. In fact a power series is determined by its value at infinitely many points. Compare this with polynomials of degree n which are uniquely determined by their value at $n + 1$ different points.

There is a second way to establish the same identity. We have

$$\begin{aligned} e^{iz} &= \cos z + i \sin z \\ e^{-iz} &= \cos z - i \sin z. \end{aligned}$$

If we multiply both equations we get

$$\begin{aligned} 1e^{i0} & \\ e^{ix} \cdot e^{-iz} & \\ (\cos z + i \sin z)(\cos z - i \sin z) & \\ \cos^2 z + \sin^2 z. & \end{aligned}$$

We now how to use the method of power series to prove

$$e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}.$$

We start with the observation that it is enough to prove the much stronger result:

$$e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$$

for any pair of complex numbers. Of course this rule is well-known for real variable

$$e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}.$$

To multiply two powers, add the exponents.

We use the same trick as before, using power series. The only complication is that now we have two variables. We fix one, lets says z_2 and we consider the other one z_1 as a variable z. We want to show

$$e^{z} \cdot e^{z_2} = e^{z+z_2}.$$

Thus is the same as an equality of power series. The problem is that to reduce to the real case, we need to assume that $z_2 = x_2$ is real. If we now put $z = x$ a real number then we get equality. So now we know

$$e^{z} \cdot e^{x_2} = e^{z+x_2},$$

where x is a real number. In particular we now know

$$e^{x}e^{iy} = e^{x+iy}.$$

One can keep going this way. The argument can be made to work but it has become somewhat absurd.