22. This and that

We give one application of the residue theorem.

Example 22.1. *Compute*

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}.$$

Note that this is a real integral and we could always compute this using trigonometric substitutions.

Here is another way, using contour integration. Consider U the open disk of radius R centred at the origin intersected with the upper half plane. The boundary of this disk is the interval $[-R, R]$ union the semicircle from R to $-R$ in the upper half plane. The function

$$f(z) = \frac{1}{1+z^2}$$

is holomorphic on the region except at i where it has a simple pole. We can apply the Residue theorem to compute the integral around the boundary. We have

$$\begin{aligned}\mathrm{Res}_i \frac{1}{1+z^2} &= \lim_{z\to i} \frac{z-i}{1+z^2} \\ &= \lim_{z\to i} \frac{1}{2z} \\ &= \frac{1}{2i} \\ &= -\frac{i}{2},\end{aligned}$$

where we applied L'Hôpital's rule to get from the first line to the second line.

Thus

$$\begin{aligned}\int_{\partial U} \frac{\mathrm{d}z}{1+z^2} &= 2\pi i \left(-\frac{i}{2}\right) \\ &= \pi,\end{aligned}$$

as long as $R > 1$ so that we capture the isolated singularity.

Now the integral around the boundary has two parts, the integral along the real axis γ_1 and the integral around the semicircle γ_2. For the integral along γ_1 we have

$$\int_{\gamma_1} \frac{\mathrm{d}z}{1+z^2} = \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^2}.$$

If we let $R \to \infty$ then we get the integral we want to compute,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} = \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{d}x}{1+x^2}.$$

For the integral along γ_2 we estimate its absolute value from above. The length L of the path we are integrating over is the length πR of a semicircle of radius R. The maximum value of the absolute value of the integrand on a circle of radius $R > 1$ is

$$\begin{aligned} \left| \frac{1}{1+z^2} \right| &= \frac{1}{|1+z^2|} \\ &\leq \frac{1}{|z^2|-1} \\ &\leq \frac{1}{R^2-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \int_{\gamma_2} \frac{\mathrm{d}z}{1+z^2} \right| &\leq LM \\ &\leq \frac{\pi R}{R^2-1}. \end{aligned}$$

As $R \to \infty$ it follows that the integral around γ_2 goes to zero. Putting all of this together yields:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2} = \pi.$$

It seems appropriate to end the course with some results about essential singularities and entire functions.

Theorem 22.2 (Casorati-Weierstrass). *Let $f(z)$ be a holomorphic function with an essential singularity at a.*

Let $b \in \mathbb{C}$. Then we may find an infinite sequence of points z_i approaching a such that $f(z_i)$ approaches b,

$$\lim_{i\to\infty} z_i = a \qquad \textit{and} \qquad \lim_{i\to\infty} f(z_i) = b.$$

Proof. It suffices to prove that for every $\delta > 0$ and $\epsilon > 0$ there is a point z in the punctured disk of radius δ about a such that $f(z)$ is in the open disk of radius ϵ around b.

Suppose that this is not the case, suppose that there is a $\delta > 0$ and an $\epsilon > 0$ such that there is no point in the punctured disk of radius δ about a mapping to a point in the open disk of radius ϵ centred around b.

Suppose that we compose f with the function

$$z \longrightarrow z - b$$

to get the function

$$z \longrightarrow f(z) - b.$$

Then we get no point in the disk of radius ϵ centred at 0. Now suppose compose with the function

$$z \longrightarrow \frac{z}{\epsilon},$$

to get the function

$$z \longrightarrow \frac{f(z) - b}{\epsilon}.$$

Now we get no point in the unit disk. Finally suppose we compose with the reciprocal function

$$z \longrightarrow \frac{1}{z}$$

to get the function

$$z \longrightarrow \frac{\epsilon}{f(z) - b} = g(z).$$

Then we get no point outside the unit disk. In other words $g(z)$ is a holomorphic function with an isolated singularity at a which is bounded, in fact bounded by 1. It follows that $g(z)$ has a removable singularity, by Riemann's theorem.

But then $g(z)$ extends naturally to a holomorphic function at a. As

$$\frac{\epsilon}{f(z) - b} = g(z)$$

it follows that

$$\epsilon = g(z)(f(z) - b),$$

so that

$$f(z) = \frac{\epsilon}{g(z)} + b.$$

As $g(z)$ is holomorphic, $f(z)$ has a pole at a, a contradiction.

Thus $f(z)$ gets arbitrarily close to b as it approaches a. □

Hopefully it is clear that holomorphic functions behave very strangely close to essential singularities.

Example 22.3. *Consider*

$$e^{-1/z^2}.$$

This is a holomorphic function with an essential singularity at zero. Perversely, we saw in the first lecture that if we restrict this function to the real axis, to get the real function

$$e^{-1/x^2}$$

the function extends to an infinitely differentiable function on the real line.

Note that the proof of (22.2) is very similar in spirit to the proof that no entire function misses an open disk. Move the centre of the disk to the origin. Rescale so the radius is one. Take reciprocals to make the inside of the disk the outside.

Now we have a bounded entire function. By Liouville's theorem it is constant. Reversing this process we must have a constant function.

It is interesting to wonder how many points an entire function can omit?

Theorem 22.4 (Little Picard theorem). *Every non-constant entire holomorphic function omits at most one value.*

Note that there are entire functions which miss one value:

Example 22.5. *The exponential function is an entire function which is never zero.*

There is a companion theorem for essential singularities:

Theorem 22.6 (Great Picard theorem). *Every function $f(z)$ with an essential singularity at a takes on every value b infinitely often arbitrarily close to a, with one possible exception for the value of b.*

Again it is easy to see the exception:

Example 22.7. *The function*

$$z \longrightarrow e^{1/z}$$

has an essential singularity at 0*.*

Note that $e^{1/z}$ close to zero records the behaviour of the exponential function close to ∞. So $e^{1/z}$ never takes on the value zero as e^w is never zero.

We can check (22.6) in this one case. Suppose that $b \neq 0$. Given any $R > 0$ we just need to find infinitely many w such that $e^w = b$ and $|w| > R$. Suppose that

$$b = re^{i\theta} \qquad \text{and} \qquad w = u + iv.$$

Then we want

$$e^u e^{iv} = re^{i\theta}$$

This says

$$e^u = r \qquad \text{and} \qquad e^{iv} = e^{i\theta}.$$

The first equation fixes u but the second equation is ambiguous, it only determines v up to 2π. Thus

$$v = \theta + 2m\pi.$$

This gives infinitely many solutions and only finitely many belong to the closed disk of radius R about 0.