21. The residue theorem

Definition 21.1. *Let f be a holomorphic function with an isolated singularity at a.*

*The **residue** of f at a, denoted $\operatorname{Res}_a f(z)$, is equal to a_{-1}, the coefficient of $1/(z-a)$ in the Laurent series expansion of f in a punctured neighbourhood of a.*

Theorem 21.2 (Residue Theorem)**.** *Let U be a bounded domain with piecewise differentiable boundary and let f be a holomorphic function on $U \cup \partial U$ except at finitely many isolated singular points $a_1, a_2, \dots, a_n$ belonging to U.*

We have

$$\int_{\partial U} f(z)\,\mathrm{d}z = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{a_i} f(z).$$

Before we prove this theorem, we give some applications.

Example 21.3. *Compute:*

$$\oint_{|z|=3} \frac{1}{(z-1)(z+1)}\,\mathrm{d}z$$

The function

$$\frac{1}{(z-1)(z+1)}$$

has isolated singularities at $z = 1$ and $z = -1$ and is otherwise holomorphic. Both of these points belong to the open unit disk of radius 3 centred at 0. The boundary of this disk is the circle of radius 3, centred at 0.

So we have to compute the residue at these two points and then apply the residue theorem. We use the method of partial fractions

$$\frac{1}{(z-1)(z+1)} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}.$$

We have

$$\operatorname{Res}_1 \frac{1}{(z-1)(z+1)} = \frac{1}{2}.$$

and

$$\operatorname{Res}_{-1} \frac{1}{(z-1)(z+1)} = -\frac{1}{2}.$$

Thus

$$\oint_{|z|=3} \frac{1}{(z-1)(z+1)}\,\mathrm{d}z = 0,$$

by the residue theorem.

To apply the residue theorem, we need to be able to compute residues efficiently. One way is to simply find the Laurent series expansion.

Example 21.4. *What is the residue of*

$$\sin\left(\frac{1}{z}\right)$$

at 0*?*

We have

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} + \dots.$$

The residue at 0 is 1, the coefficient of $1/z$.

If $f(z)$ has a simple pole, there is something easier.

Lemma 21.5. *Let f be a holomorphic function with a simple pole at a.*

Then

$$\operatorname{Res}_a f(z) = \lim_{z \to a} (z-a) f(z).$$

Proof. By assumption

$$f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots.$$

In this case

$$(z-a)f(z) = a_{-1} + a_0(z-a) + a_1(z-a)^2 + \dots.$$

We have

$$\begin{aligned}\lim_{z\to a}(z-a)f(z) &= \lim_{z\to a}(a_{-1} + a_0(z-a) + a_1(z-a)^2 + \dots) \\ &= a_{-1} \\ &= \operatorname{Res}_a f(z).\end{aligned}$$

□

Example 21.6. *Find the residues of*

$$\frac{1}{(z-1)(z+1)}.$$

As we saw before this function has simple zeroes at 1 and -1. We have

$$\begin{aligned}\operatorname{Res}_1 \frac{1}{(z-1)(z+1)} &= \lim_{z\to 1} \frac{1}{z+1} \\ &= \frac{1}{2}.\end{aligned}$$

We also have

$$\operatorname{Res}_{-1} \frac{1}{(z-1)(z+1)} = \lim_{z \to -1} \frac{1}{z-1}$$
$$= -\frac{1}{2}.$$

This is consistent with the previous results.

If $f(z)$ has a double pole, we have to be a little bit more creative:

Lemma 21.7. *Let f be a holomorphic function with a double pole at a.*

Then

$$\operatorname{Res}_a f(z) = \lim_{z \to a} ((z-a)^2 f(z))'.$$

Proof. By assumption

$$f(z) = \frac{a_{-2}}{(z-a)^2} + fin\frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots.$$

In this case

$$(z-a)^2 f(z) = a_{-2} + a_{-1}(z-a) + a_0(z-a)^2 + a_1(z-a)^3 + \dots.$$

We have

$$\begin{aligned}
\lim_{z \to a} ((z-a)^2 f(z))' &= \lim_{z \to a} (a_{-2} + a_{-1}(z-a) + a_0(z-a)^2 + a_1(z-a)^3 + \dots)' \\
&= \lim_{z \to a} (a_{-1} + 2a_0(z-a) + 3a_1(z-a)^2 + \dots)' \\
&= a_{-1} \\
&= \operatorname{Res}_a f(z).
\end{aligned}$$

□

Example 21.8. *Find the residues of*

$$\frac{e^{2z}}{z^2}.$$

This has an isolated singularity at 0, a double pole. We have

$$\begin{aligned}
\operatorname{Res}_0 \frac{e^{2z}}{z^2} &= \lim_{z \to 0} (e^{2z})' \\
&= \lim_{z \to 0} 2e^{2z} \\
&= 2.
\end{aligned}$$

There are similar results for functions with triple poles, quadruple poles. We have to multiply by a higher power of $(z-a)$, differentiate more times and divide through by an appropriate factorial.

We now turn to a proof of the residue theorem:

Proof of (21.2). Pick small disks around each singular point which are contained in U. Suppose that the radius of the disk centred at a_i is r_i. Let V be the region obtained by deleting the closed disk of radius r_i centred at a_i.

Then the boundary of V is the same as the boundary of U plus the circles of radius r_i around each a_i, but now with the opposite orientation. Note that $f(z)$ is holomorphic on V. Thus Cauchy's theorem implies:

$$\begin{aligned} 0 &= \int_{\partial V} f(z)\,\mathrm{d}z \\ &= \int_{\partial U} f(z)\,\mathrm{d}z - \sum_{i=1}^{n} \oint_{|z-a_i|=r_i} f(z)\,\mathrm{d}z. \end{aligned}$$

Now consider the integral of $f(z)$ around a typical isolated singular point a. $f(z)$ has a Laurent expansion

$$f(z) = \sum a_k (z-a)^k.$$

Then

$$\begin{aligned} \oint_{|z-a|=r} f(z)\,\mathrm{d}z &= \oint_{|z-a|=r} \sum a_k (z-a)^k\,\mathrm{d}z \\ &= \sum a_k \oint_{|z-a|=r} (z-a)^k\,\mathrm{d}z \\ &= 2\pi i a_{-1} \\ &= 2\pi i \operatorname{Res}_a f(z). \end{aligned}$$

Recall that, if m is an integer then

$$\oint_{|z-a|=r} (z-a)^m\,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } m=-1 \\ 0 & \text{otherwise.} \end{cases}$$

See lecture 19 and lecture 17, Example 17.2.

It follows that

$$\begin{aligned} \int_{\partial U} f(z)\,\mathrm{d}z &= \sum_{i=1}^{n} \oint_{|z-a_i|=r_i} f(z)\,\mathrm{d}z \\ &= 2\pi i \sum_{i=1}^{n} \operatorname{Res}_{a_i} f(z). \end{aligned}$$

□