20. Isolated Singularities

Definition 20.1. *Let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function on a region U. Let $a \notin U$.*

We say that a is an ***isolated singularity*** *of f if U contains a punctured neighbourhood of a.*

Note that $\operatorname{Log} z$ does not have an isolated singularity at 0, since we have to remove all of $(-\infty, 0]$ to get a continuous function. By contrast its derivative $1/z$ is holomorphic except at 0 and so it has an isolated singularity at 0.

Suppose that f has an isolated singularity at a. As a punctured neighbourhood of a is a special type of annulus, f has a Laurent expansion centred at a,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k,$$

valid for

$$0 < |z-a| < r,$$

for some real r.

The behaviour at a is dictated by the negative part of the Laurent expansion.

Definition 20.2. *If f has an isolated singularity at a and all of the coefficients a_k of the Laurent expansion*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k,$$

vanish if $k < 0$, then we say that f has a ***removable singularity***.

If f has a removable singularity then in fact we can extend f to a holomorphic function in a neighbourhood of a. Indeed, the Laurent expansion of f is a power series expansion, and this defines a holomorphic function in a neighbourhood of a.

Example 20.3. *The function*

$$\frac{\sin z}{z}$$

has a removable singularity at a.

Indeed,

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^3}{5!} + \dots\right)$$
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots,$$

is the Laurent series expansion of $\sin z/z$. Visibly there are no negative terms, so visibly

$$\frac{\sin z}{z}$$

extends to a holomorphic function.

Theorem 20.4 (Riemann's theorem on removable singularities)**.** *Let $f(z)$ be a holomorphic function which has an isolated singularity at a.*

Then $f(z)$ has a removable singularity at a if and only if $f(z)$ is bounded near a.

Proof. One direction is clear. If $f(z)$ is holomorphic at a then it is bounded at a.

Now suppose that $f(z)$ is bounded near a. Consider the Laurent expansion of f centred at a:

$$f(z) = \sum_k a_k (z-a)^k.$$

Note that

$$a_k = \frac{1}{2\pi i}\oint_{|z-a|=r} \frac{f(z)}{(z-a)^{k+1}}\,\mathrm{d}z,$$

for any sufficiently small circle of radius r centred at a. We have

$$|a_k| \leq LM,$$

where L is the length of the circle and M is the largest value of the absolute value of $f(z)$.

The length L of the circle is $2\pi r$. By hypothesis there is a constant M_0 such that

$$|f(z)| \leq M_0,$$

near a. Thus

$$\left|\frac{f(z)}{z^{n+1}}\right| = \frac{|f(z)|}{|z^{n+1}|}$$
$$= \frac{|f(z)|}{r^{n+1}}$$
$$\leq \frac{M_0}{r^{n+1}}.$$

(16.2) implies that

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)\,\mathrm{d}z}{z^{n+1}} \right| \\ &\leq LM \\ &\leq 2\pi r \frac{M_0}{2\pi r^{n+1}} \\ &= \frac{M_0}{r^n}. \end{aligned}$$

As r tends to zero the last quantity tends to zero if $n < 0$. The only possibility is that

$$|a_n| = 0 \qquad \text{so that} \qquad a_n = 0.$$

Thus $f(z)$ is given by a convergent power series close to a, so that f extends to a holomorphic function near a. □

Definition 20.5. *If f has an isolated singularity at a and all of the coefficients a_k of the Laurent expansion*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-a)^k,$$

vanish if $k < -n$ but $a_{-n} \neq 0$ then we say that f has a ***pole of order*** *n at a.*

Example 20.6. *The function*

$$\frac{\cos z}{z}$$

has a pole of order 1 at 0.

Theorem 20.7. *Let $f(z)$ be a holomorphic function with an isolated singularity at a.*

The following are equivalent:

1. *f has a pole of order n at a.*
2. *there is a function $g(z)$ holomorphic and non-zero at a such that*

$$f(z) = \frac{g(z)}{(z-a)^n}.$$

3. *The function*

$$\frac{1}{f(z)}$$

is holomorphic at a and has a zero of order n at a.

Proof. Suppose that (1) holds, suppose that f has a pole of order n. Then the Laurent expansion of f looks like

$$f(z) = \frac{a_{-n}}{(z-a)^n} + \frac{a_{-n+1}}{(z-a)^{n-1}} + \cdots + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \dots.$$

Let

$$g(z) = a_{-n} + a_{-n+1}(z-a) + a_{-n+2}(z-a)^2 + \dots.$$

Then g is holomorphic at a and we have

$$f(z) = \frac{g(z)}{(z-a)^n}.$$

Note that

$$g(a) = a_{-n} \neq 0.$$

Now suppose that (2) holds. Then

$$\frac{1}{f(z)} = \frac{(z-a)^n}{g(z)}.$$

As $g(a) \neq 0$ this is holomorphic at a.

Now suppose that (3) holds. Then we may write

$$\frac{1}{f(z)} = (z-a)^n g(z),$$

where $g(z)$ is holomorphic and non-zero at a. In this case

$$f(z) = \frac{1}{(z-a)^n} h(z)$$

where

$$h(z) = \frac{1}{g(z)}$$

is holomorphic at a. As $h(z)$ is holomorphic at a, it has a power series expansion

$$h(z) = \sum_{k \geq 0} a_k (z-a)^k.$$

As $h(z)$ is the reciprocal of a non-zero function $a_0 \neq 0$. Dividing through by $(z-a)^n$ we get

$$f(z) = \frac{a_0}{(z-a)^n} + \frac{a_1}{(z-a)^{n-1}} + \dots.$$

This is a Laurent series expansion starting in degree $-n$ so that f has a pole of order n. □

The final possibility for an isolated singularity is:

Definition 20.8. *Let f be a holomorphic function with an isolated singularity at a.*

*We say that a is an **essential singularity** of $f(z)$ if the Laurent series expansion has infinitely many non-zero negative terms.*

Example 20.9.

$$\sin\left(\frac{1}{z}\right)$$

has an essential singularity at 0.

Indeed

$$\sin\left(\frac{1}{z}\right) = \cdots + \frac{1}{5!z^5} - \frac{1}{3!z^3} + \frac{1}{z}.$$