2. The Argand diagram

Definition 2.1. *A **complex number** z is an expression of the form $z = x + iy$, where x and y are real numbers.*

For example

$$2 + 3i \qquad \text{and} \qquad \pi + 13i$$

are complex numbers.

Definition 2.2. *If $z = x + iy$ is a complex number, so that x and y are real numbers, x, denoted $\operatorname{Re} z$, is called the **real part** of z and y, denoted $\operatorname{Im} z$, is the called the **imaginary part**.*

Thus 2 is the real part of $2 - 3i$ and -13 is the imaginary part of $\pi - 13i$. It is customary, and certainly convenient, to visualise a complex number as being a point of the **Argand diagram**, or the **complex plane**. We put the complex number $z = x + iy$ at the point (x, y). We can think of a real number x as a complex number whose imaginary part is zero. Thus real numbers correspond to the x-axis of the Argand diagram. We say a complex number is **purely imaginary** if its real part is zero. Imaginary numbers have the form iy, where y is a real number and the purely imaginary numbers correspond to the y-axis of the Argand diagram.

Definition 2.3. *If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, their **sum**, denoted $z_1 + z_2$, is*

$$(x_1 + x_2) + i(y_1 + y_2).$$

In words, add the real and imaginary parts. For example,

$$\begin{aligned}(2 - 3i) + (\pi + 13i) &= (2 + \pi) + (-3 + 13)i \\ &= (2 + \pi) + 10i.\end{aligned}$$

In terms of the Argand diagram we just add the corresponding vectors.

Definition 2.4. *If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, their **product**, denoted $z_1 z_2$, is*

$$(x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i.$$

In practice the way to compute the product of two complex numbers is just to apply the usual rules of arithmetic and use the rule $i^2 = -1$ every time we need to get rid of a power of i. For example,

$$\begin{aligned}(2 - 3i)(\pi + 13i) &= (2\pi - (-3) \cdot 13) + (2 \cdot 13 - 3 \cdot \pi)i \\ &= (2\pi + 39) + (26 - 3\pi)i.\end{aligned}$$

One can justify all of the formal rules from the definitions (this is a somewhat tedious and uninformative exercise):

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3); \quad z_1 + z_2 = z_2 + z_1; \quad z_1 + 0 = z_1; \quad z_1 + (-z_1) = 0$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3); \quad z_1 z_2 = z_2 z_1; \quad z_1 \cdot 1 = z_1; \quad z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Here z_1, z_2 and z_2 are general complex numbers. The most interesting identity is the last one, which says you can distribute multiplication over addition, in the usual way.

Instead of using cartesian coordinates to think about complex numbers, instead one can think in terms of polar coordinates. In polar coordinates the two relevant quantities are the distance r to the origin and the angle θ to the x-axis, going counter-clockwise.

Definition 2.5. *The* ***modulus****, or* ***absolute value*** *of a complex number z, denoted $|z|$, is the distance of the point (x, y) to the origin.*

The ***argument*** *of z, denoted $\arg(z)$, is the angle the vector (x, y) makes with the x-axis.*

The argument is only well-defined up to a multiple of 2π. The **principal value** of the argument, denoted $\operatorname{Arg}(z)$, is constrained to lie in the interval $(-\pi, \pi]$. Note that

$$|z| = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \operatorname{Arg}(z) = \arctan \frac{y}{x}.$$

The modulus of $3 + 4i$ is 5 and the principal value of the argument of $1 - i$ is $-\pi/4$.

Of course we can go backwards, from polar to cartesian:

$$x = r \cos \theta \qquad \text{and} \qquad x = r \sin \theta.$$

Putting this together, we get

$$z = r \cos \theta + ir \sin \theta.$$

The most basic property of the modulus is the **triangle inequality**:

$$|z + w| \leq |z| + |w|,$$

with equality if and only if either $z = 0$ or w is a positive real scalar multiple of z.

Lemma 2.6. *If z and w are complex numbers then*

$$|z| - |w| \leq |z - w|.$$

Proof. As $z = (z - w) + w$ we have

$$|z| \leq |z - w| + |w|,$$

by the triangle inequality. Rearranging, we get

$$|z| - |w| \leq |z - w|. \qquad \square$$

Note that i is a square root of -1. $-i$ is the other square root of -1. In some sense the choice of square root is arbitrary.

Definition 2.7. *Let $z = x + iy$ be a complex number. The* ***complex conjugate****, denoted $\bar{z}$, is the complex number*

$$x - iy.$$

One can use the complex conjugate to write down a formula for the real and imaginary parts:

$$\operatorname{Re} z = \frac{z + \bar{z}}{2} \qquad \text{and} \qquad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

There are again a collection of simple identities to do with complex conjugates,

$$\overline{z + w} = \bar{z} + \bar{w}; \quad \overline{zw} = \bar{z}\bar{w}; \quad |\bar{z}| = |z|; \quad |\bar{z}|^2 = z\bar{z}.$$

The last expression gives a formula for the multiplicative inverse of a non-zero complex number

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}.$$

It follows that one can add, subtract, multiply and divide complex numbers, in the usual way.

Definition 2.8. *A complex polynomial $p(z)$ is an expression of the form*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_0, a_1, \dots, a_n$ are complex numbers.

A complex polynomial determines a complex function

$$p \colon \mathbb{C} \longrightarrow \mathbb{C},$$

in the obvious way.

Complex numbers were introduced to solve all quadratic equations. It is a truly amazing fact that in fact one can solve any complex polynomial equation:

Theorem 2.9 (Fundamental theorem of algebra)**.** *If $p(z)$ is a complex polynomial of degree $n > 0$ then $p(z)$ has a complex root, that is, there is a complex number α such that*

$$p(\alpha) = 0.$$

Remark 2.10. *One of the most striking properties of the Fundamental Theorem of Algebra is that there is no straightforward way to prove it using only algebra. Most proofs either use some analysis or some topology.*

The degree of a complex polynomial is defined in the usual way. It is equivalent to saying the function is not constant. One can use (2.9) to prove, in the usual way:

Corollary 2.11. *If $p(z)$ is a complex polynomial of degree n then there are complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ and a such that*

$$p(z) = a(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)\dots(z - \alpha_n).$$

Example 2.12. *The complex polynomial $z^2 + 1$ has roots $\pm i$.*

It follows that

$$z^2 + 1 = (z - i)(z + i),$$

which can of course be checked by hand.