19. Laurent Series

If a holomorphic function is defined on an open disk it has a power series representation on that disk. What can we say about functions holomorphic on an annulus?

Definition 19.1. *If $a \in \mathbb{C}$ and $\sigma < \rho$, belonging to $[0, \infty]$, then the region*

$$U = \{\, z \in \mathbb{C} \,|\, \sigma < |z - a| < \rho \,\}$$

is called an ***annulus****.*

In short, an annulus is the region between two circles. Note that this region is not simply connected, it has a hole in the middle. It is the simplest region not conformally equivalent to the unit disk.

Observe that there are two interesting extremes. If $\sigma = 0$ we are just excluding a. Thus we have a punctured disk. If $\rho = \infty$ we have a neighbourhood of infinity. If $\sigma = 0$ and $\rho = \infty$ then we have $U = \mathbb{C} - \{a\}$, the punctured plane.

Example 19.2. *The function*

$$z + \frac{1}{z}$$

is holomorphic on the annulus $U = \mathbb{C} - \{0\}$.

It cannot be represented by a power series, since it is not holomorphic at 0. Nor does it have a power series expansion at ∞, since it not holomorphic at ∞. Indeed

$$\begin{aligned} g(w) &= f\left(\frac{1}{w}\right) \\ &= \frac{1}{w} + w \end{aligned}$$

is not holomorphic at 0.

However it is the sum of a power series centred at 0, with radius of convergence $\rho = \infty$ and a power series expansion at ∞, with radius of convergence $1/\sigma = \infty$.

Theorem 19.3. *Let $f \colon U \longrightarrow \mathbb{C}$ be a holomorphic function on the annulus*

$$U = \{\, z \in \mathbb{C} \,|\, \sigma < |z - a| < \rho \,\}.$$

Then there are two holomorphic functions f_0 and f_∞ such that

$$f(z) = f_0(z) + f_\infty(z),$$

where $f_0(z)$ is holomorphic on the open disk centred at a of radius ρ and $f_\infty(z)$ is holomorphic outside the closed disk centred at a of radius

σ. If we require in addition that $f_\infty(z)$ is zero at infinity then $f_0(z)$ and $f_\infty(z)$ are unique with this property.

Moreover

$$f_0 = \sum_{k\geq 0} a_k(z-a)^k \qquad \text{and} \qquad f_\infty = \sum_{k<0} a_k(z-a)^k.$$

It follows that we may write

$$\begin{aligned} f(z) &= \sum_k a_k(z-a)^k \\ &= \dots + \frac{a_{-3}}{(z-a)^3} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \dots. \end{aligned}$$

where the summation is over all of the integers.

The doubly infinite series is called a Laurent series.

Example 19.4. *Consider the function*

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

This is holomorphic on the annulus

$$U = \{\, z \in \mathbb{C} \,|\, 1 < |z| < 2 \,\}.$$

Therefore it has a Laurent expansion centred at zero which converges on the annulus. To find the Laurent expansion, we use the method of partial fractions.

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

Note that this is the decomposition into a function holomorphic for $|z| < 2$ and a function holomorphic for $|z| > 1$ vanishing at infinity. We have

$$\begin{aligned} \frac{1}{z-2} &= -\frac{1}{2}\frac{1}{1-z/2} \\ &= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} + \dots. \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{1}{z-1} &= \frac{1}{z}\frac{1}{1-1/z} \\ &= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots. \end{aligned}$$

Thus

$$\frac{1}{(z-1)(z-2)} = \dots - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} + \dots.$$

Proof of (19.3). We first address uniqueness of the decomposition. Suppose that

$$f(z) = f_0(z) + f_\infty(z) = g_0(z) + g_\infty(z),$$

are two ways to write f as a combination of a holomorphic function on the open disk $|z - a| < \rho$ and on the region $|z - a| > \sigma$, where both $f_\infty(z)$ and $g_\infty(z)$ vanish at ∞.

We have

$$f_0(z) - g_0(z) = g_\infty(z) - f_\infty(z).$$

Call the common function $h(z)$. The function on the LHS is holomorphic for $|z - a| < \rho$. So $h(z)$ is holomorphic for $|z - a| < \rho$. The function on the RHS is holomorphic for $|z-a| > \sigma$ and vanishes at ∞. Therefore $h(z)$ is holomorphic for $|z - a| > \sigma$ and vanishes at ∞.

As for every complex number z we either have $|z-a| < \rho$ or $|z-a| > \sigma$ (or both, on the annulus), it follows that $h(z)$ is entire. As $h(z)$ is holomorphic at infinity, it is bounded. Therefore Liouville's theorem implies that $h(z)$ is constant. As $h(z)$ vanishes at infinity, it follows that $h(z) = 0$.

But then

$$f_0(z) = g_0(z) \qquad \text{and} \qquad g_\infty(z) = f_\infty(z).$$

This gives us uniqueness.

We now turn to existence. Pick two circles of radii

$$\sigma < s < r < \rho.$$

Cauchy's integral formula applied to (the smaller) annulus reads

$$f(z) = \frac{1}{2\pi i}\oint_{|w-a|=r} \frac{f(w)}{w-z}\,\mathrm{d}w - \frac{1}{2\pi i}\oint_{|w-a|=s} \frac{f(w)}{w-z}\,\mathrm{d}w.$$

The function

$$f_0(z) = \frac{1}{2\pi i}\oint_{|w-a|=r} \frac{f(w)}{w-z}\,\mathrm{d}w$$

is holomorphic for $|z - a| < r$ and the function

$$f_\infty(z) = \frac{1}{2\pi i}\oint_{|w-a|=s} \frac{f(w)}{w-z}\,\mathrm{d}w$$

is holomorphic for $|z-a| > s$ and tends to zero as z approaches infinity (see homework 7, problem 1, for the fact that we get holomorphic functions).

Since we can choose s and r as close to σ and ρ as we please, without changing $f_0(z)$ and $f_\infty(z)$, it follows that $f_0(z)$ is holomorphic on the open disk centred at a of radius ρ and $f_\infty(z)$ is holomorphic outside the closed disk centred at a of radius σ. □

Example 19.5. *Consider the function*

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

Now let us consider what happens when we expand it as a Laurent series centred at 1. It is not holomorphic at $z = 1$ and at $z = 2$. It is holomorphic on the annulus

$$U = \{\, z \in \mathbb{C} \,|\, 0 < |z-1| < 1 \,\}.$$

Therefore it has a Laurent expansion centred at one which converges on the annulus. As before, we write

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

Note that this is the decomposition into a function holomorphic for $|z-1| < 1$ and a function holomorphic for $|z-1| > 0$ vanishing at infinity. We have

$$\begin{aligned}\frac{1}{z-2} &= -\frac{1}{2-z} \\ &= -\frac{1}{1-(z-1)} \\ &= -1 - (z-1) - (z-1)^2 + (z-1)^3 + \dots .\end{aligned}$$

Thus

$$\frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 + (z-1)^3 + \dots .$$

We now turn to the problem of finding a formula for the coefficients of a Laurent expansion

$$f(z) = \sum_k a_k (z-a)^k.$$

Recall that, if m is an integer then

$$\oint_{|z-a|=r} (z-a)^m \,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } m = -1 \\ 0 & \text{otherwise.} \end{cases}$$

We did the case $a = 0$ and the general case is just as straightforward. Now we can compute the coefficients. If $m \geq 0$ is an integer then we have

$$\begin{aligned}
\oint_{|z|=r} f(z)(z-a)^m \,\mathrm{d}z &= \oint_{|z|=r} \left(\sum_{k=-\infty}^{\infty} a_k (z-a)^k \right) z^m \,\mathrm{d}z \\
&= \oint_{|z|=r} \sum_{k=-\infty}^{\infty} a_k (z-a)^{m+k} \,\mathrm{d}z \\
&= \sum_{k=-\infty}^{\infty} a_k \oint_{|z|=r} (z-a)^{m+k} \,\mathrm{d}z \\
&= 2\pi i a_{-1-m}.
\end{aligned}$$

since the integral on the penultimate line is non-zero only if the exponent

$$m+k=-1 \qquad \text{so that} \qquad k=-m-1$$

Thus

$$\begin{aligned}
a_k &= \frac{1}{2\pi i} \oint_{|z|=r} f(z)(z-a)^{-k-1} \,\mathrm{d}z \\
&= \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)}{(z-a)^{k+1}} \,\mathrm{d}z.
\end{aligned}$$