18. Zeroes of holomorphic functions

One of the most basic properties of polynomials $p(z)$ is that one can talk about the order of the zeroes of the polynomial. Thus $z = 0$ is a zero of order 3 of

$$p(z) = z^3(z^2 + 1).$$

One can extend this to power series, so that it makes sense to talk about the order of the zeroes of a holomorphic function:

Definition 18.1. *Let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function on a region U. We say that $a \in U$ is a* ***zero*** *of f of* ***order*** *n if all the derivatives of f up to order $n-1$ vanish at a and the nth derivative is non-zero at a.*

A zero of order one is called a **simple zero** and a zero of order two is called a **double zero**.

Lemma 18.2. *Let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function. Let $a \in U$.*

The following are equivalent:

(1) f has a zero of order n at a.

(2) f has a power series expansion centred at a of the form

$$f(z) = \sum_{k \geq n} a_k (z-a)^k = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \dots,$$

where $a_n \neq 0$.

(3) We may write

$$f(z) = (z-a)^n g(z)$$

where $g(z)$ is holomorphic at a and does not vanish at a.

Proof. Suppose that (1) holds. As f is holomorphic it has a power series

$$f(z) = \sum_{k \geq 0} a_k (z-a)^k = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \dots.$$

The mth derivative of f at a is

$$m! a_m.$$

It follows that

$$a_0 = a_1 = a_2 = \dots = a_{n-1} = 0 \qquad \text{and} \qquad a_n \neq 0.$$

Thus (2) holds.

Now suppose that (2) holds. Let

$$g(z) = a_n + a_{n+1}(z-a) + a_{n+2}(z-a)^2 + \dots.$$

It is not hard to check that the radius of convergence of the power series on the RHS is the same as the radius of convergence of the power series for f. Thus g is a holomorphic function in a neighbourhood of a. Note that

$$f(z) = (z-a)^n g(z) \qquad \text{and} \qquad g(a) = a_n \neq 0.$$

Thus (3) holds.

Finally suppose that (3) holds. If $n > 0$ then

$$\begin{aligned} f(a) &= (a-a)^n g(a) \\ &= 0. \end{aligned}$$

We have

$$f'(z) = n(z-a)^{n-1} g(z) + (z-a)^n g'(z).$$

If $n > 1$ then

$$\begin{aligned} f'(a) &= n(a-a)^{n-1} g(a) + (a-a)^n g'(a) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

In general, note that $n(z-a)^{n-1}g(z)$ has a zero of order $n-1$ by induction. Thus the first $n-2$ derivatives of $n(z-a)^{n-1}g(z)$ vanish at a and the last one does not vanish at a. On the other hand, the first $n-1$ derivatives of $(z-a)^n g'(z)$ vanish. Thus (1) holds. □

Example 18.3. *The entire function $(z-a)^n$ has a zero of order n at a.*

In this case $g(z) = 1$.

Example 18.4. *The entire function $\sin z$ has only simple zeroes.*

Indeed $\sin z$ is zero if and only if z is an integer multiple of π. The derivative of $\sin z$ is $\cos z$. This is ± 1 at the integer multiples of π. Thus $\sin z$ has only simple zeroes.

In this case

$$\begin{aligned} \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \\ &= z(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots). \end{aligned}$$

Thus

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots.$$

Note that $g(0) = 1 \neq 0$.

Example 18.5. *The entire function*

$$\cos z - 1$$

has a double zero at the even multiples of 2π*.*

Indeed the zeroes of $\cos z - 1$ are at the even multiples of 2π, since this is where $\cos z = 1$. The derivative of $\cos z$ is $-\sin z$ and this is also zero at the even multiples of 2π. The derivative of $-\sin z$, that is, the 2nd derivative of $\cos z - 1$, is $-\cos z$. This is not zero at the even multiples of 2π.

In this case

$$\begin{aligned}\cos z - 1 &= -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \\ &= z^2\left(-\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \dots\right).\end{aligned}$$

Hence

$$g(z) = -\frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \dots.$$

Note that $g(0) = -1/2 \neq 0$.

One of the most important properties of a holomorphic function is that its zeroes are isolated (assuming it is not identically zero):

Definition 18.6. *We say that a number* e *belonging to a set of complex numbers* $E \subset \mathbb{C}$ *is **isolated** if there is an open disk centred about* e *such that* e *is the only complex number in* E *belonging to the disk.*

If e is not an isolated point then we say that E is an **accumulation point** of E.

Example 18.7. *Let*

$$E = \{\, \frac{1}{n} \mid n \in \mathbb{N} \,\} \cup \{0\} \subset \mathbb{C}.$$

Every non-zero number in E is an isolated point of E. On the other hand 0 is an accumulation point, since

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Proposition 18.8. *Let* $f\colon U \longrightarrow \mathbb{C}$ *be a holomorphic function on a region* U *which is not identically zero.*

Then the zeroes of f *are isolated.*

Proof. We will assume the following result (which is true but involves a little bit of topology): f is not identically zero on any disk.

Let $a \in U$ be a zero of f. As f is holomorphic it has a power series centred at a. As f is not identically zero on this disk this power series is not identically zero. By (18.2) we may write

$$f(z) = (z - a)^n g(z),$$

where $g(z)$ is holomorphic and non-zero at a. As g is continuous it is non-zero on some disk centred at a.

Note that if b belongs to this disk and $f(b) = 0$ then $(b - a)^n = 0$ as $g(b) \neq 0$. But then $b = a$ and so a is an isolated zero of f. □

Once again this seemingly simple statement has some very strong consequences:

Proposition 18.9. *Let f and g be two holomorphic functions on the same region U.*

If the set of points E where f and g are equal contains a point a which is not isolated then $f = g$.

Proof. Let $h = f - g \colon U \longrightarrow \mathbb{C}$. Then h is a holomorphic function on U as it is the difference of two holomorphic functions. Then h is zero on E so that a is a zero of h which is not isolated.

But then h is identically zero. Thus $f - g = 0$ so that $f = g$. □

Note that $\sin z$ is zero at infinitely many points, all of the integer multiples of π. However all of those points are isolated points. The sine function is not identically zero, of course.

We return to an example we saw before:

Example 18.10.

$$\cos^2 + \sin^2 z = 1,$$

for any complex number z.

Indeed, let f be the entire function $\cos^2 z + \sin^2 z$ and let g be the constant function 1, so that g is entire. Then f and g are equal on the real line. As every point of the real line is not isolated it follows that f and g are equal.

But then

$$\cos^2 z + \sin^2 z = 1.$$