17. Power series expansion at infinity

We have already seen that entire functions are determined by what happens if $|z|$ is large, if z is going to infinity. This suggests we should explore what happens at ∞.

Definition 17.1. *We say that a function f is **holomorphic at** ∞ if the function*

$$g(w) = f\left(\frac{1}{w}\right)$$

is holomorphic at 0.

Holomorphic at 0 means that there is an open disk centred at 0 and g is holomorphic on this open disk.

In other words, to understand how $f(z)$ behaves when $z = \infty$ we make the change of variables

$$w = \frac{1}{z} \qquad \text{so that} \qquad z = \frac{1}{w}.$$

Suppose that $f(z)$ is holomorphic at ∞ then $g(w) = f(1/w)$ is holomorphic at 0 so that it has a power series expansion

$$g(w) = \sum b_n w^n = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + \dots,$$

valid for $|w| < R$, where R is the radius of convergence.

It follows that $f(z)$ has a power series expansion in descending powers of z,

$$f(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n} = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots.$$

This series converges absolutely for $|z| > 1/R$ and it converges uniformly for $|z| > r$, where $r > 1/R$.

In theory we can compute the coefficients via a line integral. We start with a simple computation, which is interesting in its own right:

Example 17.2. *If m is an integer then*

$$\oint_{|z|=r} z^m \, \mathrm{d}z = \begin{cases} 2\pi i & \textit{if } m = -1 \\ 0 & \textit{otherwise.} \end{cases}$$

There are a number of ways to see this. The first is to quote the big theorems. If $m \geq 0$ then z^m is holomorphic on closed disk of radius r centred at 0 and so the integral is zero by Cauchy's theorem. (We will say that a function f is holomorphic on a subset $E \subset \mathbb{C}$ if it is holomorphic on some open subset U containing E). If $m < 0$ then we can use Cauchy's formula. The derivative of 1 is zero and so the only thing we have to compute is when $m = -1$ and the result follows by Cauchy's integral formula.

The other is by direct computation (which is particularly easy in this case). We use the parametrisation

$$\gamma(\theta) = re^{i\theta} \qquad \text{where} \qquad \theta \in [0, 2\pi].$$

In this case we have

$$\begin{aligned}\oint_{|z|=r} z^m \, \mathrm{d}z &= \int_0^{2\pi} r^m e^{im\theta} ire^{i\theta} \, \mathrm{d}\theta \\ &= ir^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} \, \mathrm{d}\theta.\end{aligned}$$

If $m + 1 \neq 0$ it is not hard to see that the integral is zero, as $e^{i\theta}$ has period 2π. If $m + 1 = 0$ then there is no dependence on r and the integral is $2\pi i$.

Now we can compute the coefficients. If $m \geq 0$ is an integer then we have

$$\begin{aligned}\oint_{|z|=r} f(z) z^m \, \mathrm{d}z &= \oint_{|z|=r} \left(\sum_{n=0}^{\infty} \frac{b_n}{z^n} \right) z^m \, \mathrm{d}z \\ &= \oint_{|z|=r} \sum_{n=0}^{\infty} \left(\frac{b_n}{z^{n-m}} \right) \mathrm{d}z \\ &= \sum_{n=0}^{\infty} \oint_{|z|=r} \frac{b_n}{z^{n-m}} \, \mathrm{d}z \\ &= \sum_{n=0}^{\infty} b_n \oint_{|z|=r} z^{m-n} \, \mathrm{d}z \\ &= 2\pi i b_{m+1},\end{aligned}$$

since the integral on the penultimate line is non-zero only if the exponent

$$m - n = -1 \qquad \text{so that} \qquad n = m + 1.$$

Thus

$$b_n = \frac{1}{2\pi i} \oint_{|z|=r} f(z) z^{n+1} \, \mathrm{d}z.$$

Example 17.3. *The function*

$$f(z) = \frac{1}{z^n}$$

is holomorphic at ∞.

Indeed, the function

$$\begin{aligned} g(w) &= f\left(\frac{1}{w}\right) \\ &= w^n, \end{aligned}$$

is holomorphic at 0.

Example 17.4. *The function*

$$f(z) = \frac{1}{z^2+1}$$

is holomorphic at ∞.

Indeed, the function

$$\begin{aligned} g(w) &= f\left(\frac{1}{w}\right) \\ &= \frac{1}{(1/w)^2+1} \\ &= \frac{w^2}{1+w^2} \end{aligned}$$

is holomorphic at 0, as it is the quotient of two polynomials and the denominator is non-zero at 0.

As g is holomorphic at 0 it follows that $g(w)$ has a power series expansion at 0, which we can compute using the expansion of the geometric series,

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots,$$

so that

$$\begin{aligned} g(w) &= \frac{w^2}{1+w^2} \\ &= w^2 - w^4 + w^6 - w^8 + \dots. \end{aligned}$$

Thus

Example 17.5. *The Möbius transformation*

$$z \longrightarrow \frac{az+b}{cz+d} = f(z),$$

where $ab - bc \neq 0$ is holomorphic at ∞ if and only if $c \neq 0$.

Indeed,

$$\begin{aligned} g(w) &= f\left(\frac{1}{w}\right) \\ &= \frac{a(1/w)+b}{c(1/w)+d} \\ &= \frac{a+bw}{c+dw} \end{aligned}$$

is holomorphic at 0 if and only if $c \neq 0$.