16. Liouvilles theorem

To apply Cauchy's formula we will need some easy estimates.

Definition 16.1. *Let*

$$\gamma\colon [\alpha,\beta] \longrightarrow \mathbb{C},$$

be a differentiable curve. The ***length*** *of γ is the integral*

$$L = \int_{\alpha}^{\beta} (x'(t)^2 + y'(t))^{1/2}\,\mathrm{d}t.$$

If one picks points $a = z_0, z_1, \dots, z_n = b$, where $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ then the distance from z_i to z_{i+1} along γ is approximated by the length of the line connecting z_i to z_{i+1}, which is

$$((x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2)^{1/2}$$

by Pythagoras. By the mean value theorem we can find τ_i and υ_i in the interval (t_i, t_{i+1}) such that

$$x_{i+1} - x_i = x'(\tau_i)(t_{i+1} - t_i) \qquad \text{and} \qquad y_{i+1} - y_i = y'(\upsilon_i)(t_{i+1} - t_i).$$

Thus the length of the line connecting z_i to z_{i+1} is

$$\begin{aligned}((x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2)^{1/2} &= ((x'(\tau_i)(t_{i+1} - t_i))^2 + (y'(\upsilon_i)(t_{i+1} - t_i))^2)^{1/2} \\ &= ((x'(\tau_i))^2 + (y'(\upsilon_i))^2)^{1/2}(t_{i+1} - t_i).\end{aligned}$$

Summing over i we get a Riemann sum approximating the integral in (16.1).

Note that

$$(x'(t)^2 + y'(t))^{1/2} = |\gamma'(t)|$$

is the length of the tangent vector γ at t. Thus the length is also

$$L = \int_{\alpha}^{\beta} |\gamma'(t)|\,\mathrm{d}t,$$

the integral of the speed.

If γ is piecewise differentiable, we can define the length by simply adding together the lengths of the differentiable pieces.

If

$$\gamma\colon [\alpha,\beta] \longrightarrow U$$

is a curve and $f\colon U \longrightarrow \mathbb{C}$ is continuous then M denotes the maximum value of the absolute value of f over the curve γ:

$$M = \sup_{t\in[\alpha,\beta]} |f(\gamma(t))|.$$

We have the following basic but very useful:

Lemma 16.2. *Let $f\colon U \longrightarrow \mathbb{C}$ be a continuous function over a region U and let*

$$\gamma\colon [\alpha, \beta] \longrightarrow U,$$

be a piecewise differentiable curve.

Then

$$\left| \int_\gamma f(z)\, \mathrm{d}z \right| \leq LM.$$

It is easy to check (16.2) by using Riemann sums and the triangle inequality.

Theorem 16.3 (Liouville's theorem)**.** *Every bounded entire function is constant.*

Proof. By assumption there is a real number M_0 such that

$$|f(z)| \leq M_0.$$

As $f(z)$ is entire it has a power series expansion whose radius of convergence is ∞,

$$f(z) = \sum_n a_n z^n.$$

The coefficients are given by Cauchy's formula

$$a_n = \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)\, \mathrm{d}z}{z^{n+1}},$$

where the radius is any positive real number r. We estimate the absolute value of a_n.

The circle of radius r centred at the origin has length

$$L = 2\pi r.$$

We also have

$$\begin{aligned} \left| \frac{f(z)}{z^{n+1}} \right| &= \frac{|f(z)|}{|z^{n+1}|} \\ &= \frac{|f(z)|}{r^{n+1}} \\ &\leq \frac{M_0}{r^{n+1}}. \end{aligned}$$

(16.2) implies that

$$\begin{aligned}
|a_n| &= \left| \frac{1}{2\pi i} \oint_{|z|=r} \frac{f(z)\,\mathrm{d}z}{z^{n+1}} \right| \\
&\leq LM \\
&\leq 2\pi r \frac{M_0}{2\pi r^{n+1}} \\
&= \frac{M_0}{r^n}.
\end{aligned}$$

As r tends to infinity the last quantity tends to zero if $n > 0$. The only possibility is that

$$|a_n| = 0 \qquad \text{so that} \qquad a_n = 0.$$

Thus

$$f(z) = a_0$$

is a constant. □

The inequality

$$|a_n| r^n \leq \sup_{|z|=r} |f(z)|$$

is sometimes known as **Cauchy's inequality**.

It is convenient to introduce the notion of the limit at ∞. One common trick in real variables is to use the fact that a function $h(x)$ tends to infinity if and only if $1/h(x)$ tends to infinity. We can do the same thing in complex variable but now for the input as well as the output:

Definition 16.4. *Let $U \subset \mathbb{C}$ be a region. We say that U is a neighbourhood of ∞ if there is a real number R such that if $|z| > R$ then $z \in U$.*

Let $f\colon U \longrightarrow \mathbb{C}$ be a function defined on a region U which is a neighbourhood of infinity. The limit of $f(z)$ as z goes to infinity is

$$\lim_{z\to\infty} f(z) = \lim_{w\to 0} f\left(\frac{1}{w}\right).$$

Note that w tends towards zero if and only if $|w|$ tends towards zero if and only if $|z|$ tends towards ∞. Note also that U is a neighbourhood of infinity if and only if the image of U under the reciprocal map contains an open disk centred at the origin.

Theorem 16.5 (Fundamental theorem of algebra)**.** *If $p(z)$ is a complex polynomial of degree $n > 0$ then $p(z)$ has a complex root, that is, there is a complex number α such that*

$$p(\alpha) = 0.$$

Proof. Suppose that $p(z)$ is a polynomial with no roots. We are going to show that $p(z)$ has degree zero.

Let

$$f(z) = \frac{1}{p(z)}.$$

As we are assuming that $p(z)$ is never zero, it follows that $f(z)$ is entire.

Suppose that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_n \neq 0$. There is no harm in dividing through by a_n so that $a_n = 1$. Consider

$$\frac{p(z)}{z^n} = 1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}.$$

As z goes to infinity this tends to 1. Thus $|p(z)|$ is bounded away from zero and so $|f(z)|$ is bounded from above. But then f is constant by Liouville's theorem so that $p(z)$ is constant. It follows that the degree of $p(z)$ is zero. $\square$