15. Cauchys integral formula

Theorem 15.1 (Cauchy's Integral formula)**.** *Let U be a bounded region with piecewise smooth boundary ∂U. Let $a \in U$.*

If $f(z)$ has continuous partial derivatives on some open subset $V \supset U \cup \partial U$ and the real and imaginary parts of f satisfy the Cauchy-Riemann equations then

$$f(a) = \frac{1}{2\pi i} \oint_{\partial U} \frac{f(z)}{z-a} \mathrm{d}z.$$

Proof. As U is open, we may pick a closed disk centred at a contained in U. Suppose that the radius of this disk is $\epsilon > 0$. Let U_ϵ be the region obtained by deleting the closed disk of radius ϵ centred at a.

Then the boundary of U_ϵ is equal to the boundary of U plus the boundary of the open disk of radius ϵ centred at a, namely the circle of radius ϵ centred at a, but with the reverse orientation. Let γ be this boundary circle traversed in the counterclockwise direction.

Note that the function

$$\frac{f(z)}{z-a}$$

is holomorphic on U_ϵ. Therefore by Cauchy's theorem we have

$$\begin{aligned} \int_{\partial U} \frac{f(z)}{z-a} \mathrm{d}z - \int_{\gamma} \frac{f(z)}{z-a} \mathrm{d}z &= \int_{\partial U - \gamma} \frac{f(z)}{z-a} \mathrm{d}z \\ &= \int_{\partial U_\epsilon} \frac{f(z)}{z-a} \mathrm{d}z \\ &= 0. \end{aligned}$$

It follows then that

$$\int_{\partial U} \frac{f(z)}{z-a} \mathrm{d}z = \oint_{|z-a|=\epsilon} \frac{f(z)}{z-a} \mathrm{d}z.$$

Note that the LHS is independent of the radius of the circle. So we are reduced to showing the result when U is an open disk centred at a of any radius ϵ contained in V.

We calculate the integral on the RHS using the following parametrisation:

$$\gamma(\theta) = a + \epsilon e^{i\theta} \qquad \text{where} \qquad \theta \in [0, 2\pi].$$

We have

$$\begin{aligned} \frac{\mathrm{d}z}{z-a} &= \frac{i\epsilon e^{i\theta} \mathrm{d}\theta}{\epsilon e^{i\theta}} \\ &= i \mathrm{d}\theta. \end{aligned}$$

Thus

$$\oint_{|z-a|=\epsilon} \frac{f(z)}{z-a} \mathrm{d}z = i \int_0^{2\pi} f(a+\epsilon e^{i\theta})\, \mathrm{d}\theta.$$

To calculate the integral on the RHS we use the fact that it is independent of ϵ. We have

$$\frac{1}{2\pi} \int_0^{2\pi} f(a+\epsilon e^{i\theta})\, \mathrm{d}\theta = f(a) + \frac{1}{2\pi} \int_0^{2\pi} \left[f(a+\epsilon e^{i\theta}) - f(a) \right] \, \mathrm{d}\theta$$

It remains to show that the last integral is zero. As f has continuous partial derivatives, it is certainly continuous. Thus $f(a+\epsilon e^{i\theta})$ tends uniformly to $f(a)$ as ϵ goes to zero. Thus the integral

$$\frac{1}{2\pi} \int_0^{2\pi} \left[f(a+\epsilon e^{i\theta}) - f(a) \right] \, \mathrm{d}\theta$$

tends to zero as ϵ tends to zero. As the integral is independent of ϵ the only possibility is that it is zero to begin with. □

Theorem 15.2. *Let $f \colon U \longrightarrow \mathbb{C}$ be a function on a region whose real and imaginary parts have continuous partial derivatives.*

The following are equivalent:

(1) *the real and imaginary parts of f satisfy the Cauchy-Riemann equations.*
(2) *f is analytic.*
(3) *f is holomorphic.*

Proof. We have already seen that if f is analytic then it is holomorphic and we have already seen that if f is holomorphic then the real and imaginary parts of f satisfy the Cauchy-Riemann equations.

It remains to show that if the the real and imaginary parts of f satisfy the Cauchy-Riemann equations then f is analytic. Pick a point $a \in U$ and pick a closed disk contained in U centred at a. Let γ be the boundary of this closed disk traversed in the counterclockwise direction. If z belongs the open disk bounded by γ then Cauchy's integral formula reads

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} \mathrm{d}w.$$

We have

$$\begin{aligned}\frac{1}{w-z} &= \frac{1}{w-a-(z-a)} \\ &= \frac{1}{w-a}\frac{1}{1-\frac{(z-a)}{w-a}} \\ &= \frac{1}{w-a} + \frac{(z-a)}{(w-a)^2} + \frac{(z-a)^2}{(w-a)^3} + \dots.\end{aligned}$$

We consider this as a power series in z centred at a. We have uniform convergence when the absolute value of the geometric ratio

$$\left|\frac{z-a}{w-a}\right| < 1.$$

As $|w-a|$ is a constant, we therefore have uniform convergence if we stay away from γ. Therefore we can integrate the power series term by term:

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_\gamma \left(\frac{f(w)}{w-a} + \frac{(z-a)f(w)}{(w-a)^2} + \frac{(z-a)^2 f(w)}{(w-a)^3} + \dots\right)\mathrm{d}w \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,\mathrm{d}w + \frac{1}{2\pi i}\int_\gamma \frac{(z-a)f(w)}{(w-a)^2}\,\mathrm{d}w + \frac{1}{2\pi i}\int_\gamma \frac{(z-a)^2 f(w)}{(w-a)^3}\,\mathrm{d}w + \dots \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,\mathrm{d}w + \frac{(z-a)}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^2}\,\mathrm{d}w + \frac{(z-a)^2}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^3}\,\mathrm{d}w + \dots \\ &= a_0 + a_1(z-a) + a_2(z-a)^2 + \dots,\end{aligned}$$

where

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,\mathrm{d}w.$$

It follows that $f(z)$ is analytic. □

Note that we can extract a little bit more from the proof.

Theorem 15.3. *Let $f\colon U \longrightarrow \mathbb{C}$ be a holomorphic function on a region U.*

If $a \in U$ then we can write

$$f(z) = \sum a_n(z-a)^n$$

where the radius of convergence is at least the radius of any open disk centred at a contained in U, that is, at least the distance of a to the *closest point on the boundary. Further*

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$$

and the nth derivative of f at a is given by

$$\frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z.$$

Proof. The first two statements are immediate from the proof of (15.2).

The last statement follows from the fact that the nth derivative of f at a is equal to

$$n! a_n.$$

□

The last formula for the derivatives of f is also known as Cauchy's formula.

Corollary 15.4. *Let $f \colon U \longrightarrow \mathbb{C}$ be a holomorphic function such that the real and imaginary parts of f have continuous partial derivatives.*

Then f is infinitely differentiable.

Proof. By (15.2) f is analytic. But analytic functions are infinitely differentiable. □