14. Cauchys Theorem

We want to extend the definition of a line integral to the complex case. If we have a path γ we simply make the following definition:

$$\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y.$$

and use this to define the line integral in the natural way:

$$\int_\gamma h(z)\,\mathrm{d}z = \int_\gamma h(z)\,\mathrm{d}x + i\int_\gamma h(z)\,\mathrm{d}y.$$

Note that if you break a line integral into pieces one can think of the piece of curve from z_i to z_{i+1} as being approximated by

$$\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y.$$

Example 14.1. *Let us compute*

$$\int_\gamma z^2\,\mathrm{d}z,$$

where γ is the straight line segment from 0 to $1+i$.

We use the parametrisation

$$\gamma(t) = t + it \qquad \text{where} \qquad t \in [0,1].$$

In this case

$$\begin{aligned}\mathrm{d}z &= \mathrm{d}x + i\,\mathrm{d}y \\ &= \mathrm{d}t + i\,\mathrm{d}t \\ &= (1+i)\mathrm{d}t.\end{aligned}$$

Thus

$$\begin{aligned}\int_\gamma z^2\,\mathrm{d}z &= \int_0^1 (t+it)^2(1+i)\,\mathrm{d}t \\ &= \int_0^1 t^2(1+i)^3\,\mathrm{d}t \\ &= (1+i)^3\left[\frac{t^3}{3}\right]_0^1 \\ &= \frac{(1+i)^3}{3} \\ &= -\frac{2}{3}(1-i).\end{aligned}$$

Example 14.2. *Let us compute*

$$\oint_{|z|=1} \frac{\mathrm{d}z}{z}.$$

Here the circle around the integral indicates we are integrating around a closed path and that we are traversing the path so that the interior is on the left. In practice this often means we go counterclockwise.

We use the parametrisation

$$\gamma(t) = e^{it} \qquad \text{where} \qquad t \in [0, 2\pi].$$

We have

$$\begin{aligned} \frac{\mathrm{d}z}{z} &= \frac{ie^{it}\mathrm{d}t}{e^{it}} \\ &= i\,\mathrm{d}t. \end{aligned}$$

It follows that

$$\begin{aligned} \oint_{|z|=1} \frac{\mathrm{d}z}{z} &= \int_0^{2\pi} i\,\mathrm{d}t \\ &= 2\pi i. \end{aligned}$$

Theorem 14.3 (Cauchy's Theorem)**.** *Let V be a region and let U be a bounded open subset whose boundary is the finite union of continuous piecewise smooth paths such that $U \cup \partial U \subset V$.*

If the real and imaginary parts of the function $f\colon V \longrightarrow \mathbb{C}$ have continuous partial derivatives and they satisfy the Cauchy Riemann equations then

$$\int_{\partial U} f(z)\,\mathrm{d}z = 0.$$

Proof. We have

$$\begin{aligned} \int_{\partial U} f(z)\,\mathrm{d}z &= \int_{\partial U} (u + iv)\,\mathrm{d}(x + iy) \\ &= \int_{\partial U} u\,\mathrm{d}x - \int_{\partial U} v\,\mathrm{d}y + i\left(\int_{\partial U} v\,\mathrm{d}x + \int_{\partial U} u\,\mathrm{d}y\right) \\ &= \iint_U \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x\mathrm{d}y + i\iint_U \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) \mathrm{d}x\mathrm{d}y \\ &= \iint_U 0\,\mathrm{d}x\mathrm{d}y + i\iint_U 0\,\mathrm{d}x\mathrm{d}y \\ &= 0. \end{aligned}$$

□

Example 14.4. *Consider*

$$\oint_{(x-1)^2/2+(y-2)^2/3=1} (z+3)e^{z^2-5z+6}\,\mathrm{d}z.$$

The integrand

$$(z+3)e^{z^2-5z+6}$$

is entire, as it is the product of a polynomial and the exponential of a polynomial. The curve is an ellipse, and it bounds the interior of the ellipse. Therefore Cauchy's theorem implies that the line integral is zero.