13. Line integrals and Greens theorem

We are going to integrate complex valued functions f over paths γ in the Argand diagram. Typically the paths are continuous piecewise differentiable paths. For example, the sides of a rectangle.

γ is a parametrised path but the value of the integral will be independent of the parametrisation, it will only depend on the image curve.

Suppose we are given continuous complex valued functions $P\colon U \longrightarrow \mathbb{C}$ and $Q\colon U \longrightarrow \mathbb{C}$ defined on some region U and let γ be a differentiable path in U, $\gamma\colon [\alpha, \beta] \longrightarrow U$ from $a = \gamma(\alpha)$ to $b = \gamma(\beta)$. Pick points $z_0, z_1, \dots, z_n$ on the path, where $z_0 = a$ and $z_n = b$. We define the line integral

$$\int_\gamma P \,\mathrm{d}x + Q \,\mathrm{d}y$$

to be the limit of the Riemann sums

$$\sum P(z_i)(x_{i+1} - x_i) + Q(z_i)(y_{i+1} - y_i),$$

as the distance between successive points goes to zero. The limit exists, as P and Q are continuous.

Suppose that the point $z_i = \gamma(t_i)$, where

$$\alpha = t_0 < t_1 < \dots t_{n-1} < t_n = \beta.$$

By the mean value theorem, we can find $\tau_i \in (t_i, t_{i+1})$ such that

$$x(t_{i+1}) - x(t_i) = x'(\tau_i)(t_{i+1} - t_i).$$

It follows that

$$\sum P(z_i)(x_{i+1} - x_i) = \sum P(\gamma(t_i))x'(\tau_i)(t_{i+1} - t_i).$$

Now the RHS is a Riemann sum for the integral

$$\int_\alpha^\beta P(\gamma(t))x'(t)\,\mathrm{d}t.$$

Putting all of this together we get

$$\int_\gamma P \,\mathrm{d}x + Q \,\mathrm{d}y = \int_\alpha^\beta P(\gamma(t))\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t + \int_\alpha^\beta Q(\gamma(t))\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t.$$

Thus to evaluate the line integral on the LHS, we just pick a parametrisation and evaluate the RHS.

Example 13.1. *Suppose we want to evaluate*

$$\int_\gamma xy \,\mathrm{d}x,$$

where γ goes around the quarter unit circle in the first quadrant, starting and ending at the origin.

Note we usually orient γ so that the interior is on the left.

First observe that γ is a continuous piecewise differentiable curve, with three parts, γ_1, γ_2 and γ_3. The horizontal line segment from 0 to 1, the arc of the circle from 1 to i and the vertical line segment from i down to 0.

A natural parametrisation for γ_1 is

$$\gamma_1(t) = t,$$

on the interval $[0, 1]$. In this case $x'(t) = 1$ and $y'(t) = 0$. As xy is zero on γ_1 we have

$$\int_{\gamma_1} xy \, \mathrm{d}x = \int_0^1 t \cdot 0 \cdot 1 \, \mathrm{d}t = 0.$$

A natural parametrisation for γ_2 is given by

$$\gamma_2(t) = e^{it} = \cos t + i \sin t,$$

on the interval $[0, \pi/2]$. In this case

$$\gamma_2'(t) = -\sin t + i \cos t.$$

Thus

$$\begin{aligned}
\int_{\gamma_2} xy \, \mathrm{d}x &= \int_0^{\pi/2} \cos t \cdot \sin t \cdot - \sin t \, \mathrm{d}t \\
&= - \int_0^{\pi/2} \cos t \sin^2 t \, \mathrm{d}t \\
&= -\frac{1}{3} \left[\sin^3 t \right]_0^{\pi/2} \\
&= -\frac{1}{3}.
\end{aligned}$$

A natural parametrisation for γ_3 is

$$\gamma_3(t) = i(1 - t),$$

on the interval $[0, 1]$. In this case $x'(t) = 0$ and $y'(t) = -1$. As xy is zero on γ_3 we have

$$\int_{\gamma_3} xy \, \mathrm{d}x = \int_0^1 i(1 - t) \cdot 0 \cdot 1 \, \mathrm{d}t = 0.$$

Putting all of this together we get

$$\begin{aligned}\int_{\gamma} xy \, \mathrm{d}x &= \int_{\gamma_1} xy \, \mathrm{d}x + \int_{\gamma_2} xy \, \mathrm{d}x + \int_{\gamma_3} xy \, \mathrm{d}x \\ &= 0 - \frac{1}{3} + 0 \\ &= -\frac{1}{3}.\end{aligned}$$

Theorem 13.2 (Green's theorem)**.** *Let U be a bounded region whose boundary ∂U is a finite union of continuous piecewise differentiable curves. Let P and Q be two functions which have continuous partial derivatives on some region V containing $U \cup \partial U$.*

Then

$$\int_{\partial U} P \, \mathrm{d}x + Q \, \mathrm{d}y = \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y.$$

Note that we orient ∂U such that the region U is on the left.

Note the way we have stated (13.2) there is no possibility of giving a proof in this class; there are serious issues coming from topology which first need to be addressed. In practice, if we apply (13.2) to disks or squares or to any concrete region these issues disappear.

To illustrate how to use Green's theorem, let us go back to the example above. In this case the region U is the intersection of the unit disc and the first quadrant (not including the axes). The boundary ∂U is then the curve γ. $P(x, y) = xy$ has continuous partial derivatives on the whole of $\mathbb{C}$. So we can apply Green's theorem.

We compute the RHS

$$
\begin{aligned}
\iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y &= \iint_U -x \, \mathrm{d}x\mathrm{d}y \\
&= \iint_U -r\cos\theta r \, \mathrm{d}r\mathrm{d}\theta \\
&= -\int_0^{\pi/2} \int_0^1 r^2 \cos\theta \, \mathrm{d}r\mathrm{d}\theta \\
&= -\frac{1}{3} \int_0^{\pi/2} \Big[r^3 \cos\theta \Big]_0^1 \mathrm{d}\theta \\
&= -\frac{1}{3} \int_0^{\pi/2} \cos\theta \, \mathrm{d}\theta \\
&= -\frac{1}{3} \Big[\sin\theta \Big]_0^{\pi/2} \\
&= -\frac{1}{3},
\end{aligned}
$$

as expected.