12. Conformal maps

Let

$$\gamma\colon (-\epsilon, \epsilon) \longrightarrow U$$

be a differentiable curve in a region U. Let $a = \gamma(0) \in U$.

Definition 12.1. *The* ***tangent vector*** *to the curve γ at the point a is*

$$\gamma'(0).$$

Note that if $\gamma(t) = x(t) + iy(t)$ then

$$\begin{aligned}\gamma'(0) &= \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t} \\ &= x'(0) + iy'(0).\end{aligned}$$

Proposition 12.2. *If γ is a differentiable curve and f is holomorphic on U then the tangent vector to the curve $f \circ \gamma$ at the point $b = f(a)$ is the vector*

$$(f \circ \gamma)'(0) = f'(a)\gamma'(0).$$

Proof. This a variant on the chain rule. Suppose that $\gamma'(0) \neq 0$.

We have

$$\begin{aligned}(f \circ \gamma)'(0) &= \lim_{t\to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \\ &= \lim_{t\to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \frac{\gamma(t) - \gamma(0)}{t} \\ &= \lim_{t\to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{\gamma(t) - \gamma(0)} \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t} \\ &= f'(a)\gamma'(0).\end{aligned}$$

If $\gamma'(0) = 0$ it is not hard to check that $(f \circ \gamma)'(0)$ is also zero and the formula is still correct. □

Even though the formula (12.2) looks quite innocuous it has a very striking consequence. We can think of the tangent vector as starting at the point $a = \gamma(0)$ and it points in the direction of the tangent line.

Composing with f moves the starting point to $b = f(a)$. On the other hand, multiplying by a complex number $f'(a)$ rescales by the magnitude and rotates through the argument. So composing with $f(z)$ rotates all tangent vectors at a to all differentiable curves through a by the same angle.

Definition 12.3. *We say that $g\colon U \longrightarrow \mathbb{C}$ is* ***conformal*** *at a point $a \in U$ if for any two curves γ_1 and γ_2 such that $\gamma_1(0) = \gamma_2(0) = a$,* *with non-zero tangents at a, the two curves $g \circ \gamma_1$ and $g \circ \gamma_2$ have non-zero tangents and the angle between the tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$ is the same as the angle between the tangent vectors $(g \circ \gamma_1)'(0)$ and $(g \circ \gamma_2)'(0)$.*

We say that g is a ***conformal map*** *between two regions U and V if g is a bijection between U and V and g is conformal at every point of U.*

In short, a conformal map preserves angles. Note that complex conjugation $z \longrightarrow \bar{z}$ is almost a conformal map. It preserves the magnitude of the angle but it changes the orientation.

Theorem 12.4. *Let $f \colon U \longrightarrow V$ be a bijection between two regions.*

Then f is holomorphic, with nowhere vanishing derivative, if and only if it is conformal.

We have already seen that if f is holomorphic and the derivative is nowhere zero then it is conformal. The converse is quite striking but actually not so hard to prove.

The most straightforward way to check a map is a bijection is to write down the inverse map.

Example 12.5. *The map $z \longrightarrow z^2$ is a bijection of the upper half plane with the region $\mathbb{C} \setminus [0, \infty)$.*

As $z \longrightarrow z^2$ is holomorphic and the derivative is nowhere zero, it follows that this map preserves angles. If we write

$$f(z) = u(x, y) + iv(x, y) = x^2 - y^2 + i(2xy)$$

then the level curves of u and v are orthogonal in (x, y)-plane, since the curves $u = \text{cst}$ and $v = \text{cst}$ are orthogonal in the (u, v) plane.

The map $z \longrightarrow e^z$ is entire and the derivative is nowhere zero, since e^z is nowhere zero. Thus the exponential map is locally everywhere conformal. The image of a horizontal line is a half line through the origin and the image of a vertical line is a circle centred at the origin. As expected these are orthogonal.

The fact that holomorphic maps are conformal is very useful in the design of airplane wings. If you take a cross-section of an airplane wing you get a region in the plane. The performance of the airplane wing is determined by how air flows around the wing.

To determine this, we need to a solve a system of PDE's. To attack this problem directly is quite hard. However, what is important is only how the air flows over the wing and the only important feature is the angle of attack.

The idea is to solve the PDE for one fixed choice of cross-section. The easiest to solve is the unit disk, since in this case one can exploit the symmetry of the unit disk.

Now the problem is to transform an arbitrary region to the unit disk using a conformal map, or what comes to the same thing, a holomorphic map.

So now the question is which regions U are conformally equivalent to the open unit disk Δ? At this point topology comes into the picture.

If $X \subset \mathbb{C}$ is a subset then we say that X is **simply connected** if X is path connected and every closed path can be continuously deformed to a constant map, keeping the endpoints fixed (actually this is equivalent to allowing the endpoints to move, as long as the path is closed). Informally, think of the closed path as a rubber band. Can you move the rubber band around until it shrinks to a point?

Open and closed disks are simply connected and so is the upper half plane and angular regions. An annulus is not simply connected. It is not hard to check that if U is conformally equivalent to the unit disk then U has to be simply connected.

Once again this easy necessary condition is in fact sufficient:

Theorem 12.6 (Riemann mapping theorem)**.** *Every simply connected region, except the entire complex plane $\mathbb{C}$, is conformally equivalent to the unit disk Δ.*

We will see later that the unit disk and the whole complex plane are not conformally equivalent.

We have already seen many instances of (12.6) in the lectures and in the homework.

Example 12.7. *Let*

$$U = \{ z \in \mathbb{C} \mid \alpha < \arg(z) < \beta \}$$

be an angular wedge.

Then U is simply connected. How can we map this conformally to the unit disk? First note that we can do this in stages. The first stage is to make the first angle $\alpha = 0$. This is easy, multiply by $e^{-i\alpha}$,

$$z \longrightarrow e^{-i\alpha} z.$$

This has the effect of rotating the complex plane through an angle of $-\alpha$. So we are reduced to the situation

$$U = \{ z \in \mathbb{C} \mid 0 < \arg(z) < \beta \}.$$

The next thing is make $\beta = \pi$. For this we use a power function,

$$z \longrightarrow z^p,$$

where p is a positive real number. As always, we have to make sense of the ambiguity and as always we use the principal value of the logarithm. If

$$w = z^p \qquad \text{then} \qquad \log w = p \log z.$$

If we choose the principal value of the logarithm then we get

$$w = e^{p \operatorname{Log} z}.$$

Thus we get a well-defined holomorphic map with nowhere vanishing derivative on $\mathbb{C} \setminus (-\infty, 0]$.

Now we simply take

$$p = \frac{\pi}{\beta}.$$

Thus we reduce to the case where $\alpha = 0$ and $\beta = \pi$, which is the upper half plane $\mathbb{H}$. Finally we map this to the unit disk Δ using a Möbius transformation. We have to map the real line to the unit circle. We map 0 to -1, 1 to 1 and ∞ to i and then invert if we have to.

It is possible but somewhat involved to make the definition of simply connected formal. Fortunately for the complex plane there is an ad hoc way to get around this.

Definition 12.8. *We say that an open subset $U \subset \mathbb{C}$ is **simply connected** if U is path connected and the complement inside the extended complex plane is connected.*

To make sense of being connected in the extended complex plane we use the Riemann sphere.

For example the complement of an annulus has two connected components the smaller disk and the complement of the bigger disk.