11. The Cauchy-Riemann equations

Let $f\colon U \longrightarrow \mathbb{C}$ be a function. To say that f is differentiable at a point $a \in U$ means that the limit

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists.

We can try to simplify this two dimensional picture to a one dimensional picture by considering what happens as you approach a along a curve. There are many curves one can try but the most obvious curves are the horizontal line through a and the vertical line through a.

The horizontal line through a is defined by the condition that y is constant. If $a = a_0 + ia_1$ then we have the line $y = a_1$. The curve

$$\gamma(t) = (a_0 + t) + ia_1 = a + t$$

sends (a piece of) the real line to the horizontal line through a in the region U. Note that 0 gets sent to a, that is, $\gamma(0) = a$. If f is differentiable at a we have

$$\lim_{t \to 0} \frac{f(a + t) - f(a)}{t} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a},$$

since approaching along a horizontal line is one way to compute the limit on the RHS.

Similarly the vertical line through a is defined by the condition that x is constant. As $a = a_0 + ia_1$ we have the line $x = a_0$. The curve

$$\gamma(t) = a_0 + i(a_1 + t) = a + ti$$

sends (a piece of) the real line to the vertical line in the region U. Note that 0 gets sent to a, that is, $\gamma(0) = a$. If f is differentiable at a we have

$$\lim_{t \to 0} \frac{f(a + ti) - f(a)}{it} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a},$$

since approaching along a vertical line is one way to compute the limit on the RHS.

On the other hand, now suppose that we decompose f into its real and imaginary parts:

$$f(z) = u(x, y) + iv(x, y).$$

Then the horizontal limit becomes

$$\begin{aligned}\lim_{t\to 0}\frac{f(a+t)-f(a)}{t} &= \lim_{t\to 0}\frac{u(a_0+t,a_1)+iv(a_0+t,a_1)-u(a_0,a_1)-iv(a_0,a_1)}{t}\\ &= \lim_{t\to 0}\frac{u(a_0+t,a_1)-u(a_0,a_1)}{t}+\lim_{t\to 0}\frac{iv(a_0+t,a_1)-iv(a_0,a_1)}{t}\\ &= \lim_{t\to 0}\frac{u(a_0+t,a_1)-u(a_0,a_1)}{t}+i\lim_{t\to 0}\frac{v(a_0+t,a_1)-v(a_0,a_1)}{t}\\ &= \left.\frac{\partial u}{\partial x}\right|_{(a_0,a_1)}+i\left.\frac{\partial v}{\partial x}\right|_{(a_0,a_1)}.\end{aligned}$$

On the other hand the vertical limit becomes

$$\begin{aligned}\lim_{t\to 0}\frac{f(a+ti)-f(a)}{it} &= -i\lim_{t\to 0}\frac{u(a_0,a_1+t)+iv(a_0,a_1+t)-u(a_0,a_1)-iv(a_0,a_1)}{t}\\ &= -i\lim_{t\to 0}\frac{u(a_0,a_1+t)-u(a_0,a_1)}{t}-i\lim_{t\to 0}\frac{iv(a_0,a_1+t)-iv(a_0,a_1)}{t}\\ &= \lim_{t\to 0}\frac{v(a_0+t,a_1)-v(a_0,a_1)}{t}-i\lim_{t\to 0}\frac{u(a_0+t,a_1)-u(a_0,a_1)}{t}\\ &= \left.\frac{\partial v}{\partial y}\right|_{(a_0,a_1)}-i\left.\frac{\partial u}{\partial y}\right|_{(a_0,a_1)}.\end{aligned}$$

For the derivative of f to exist at a we must have that both limits are the same. Equating real and imaginary parts gives

$$\left.\frac{\partial u}{\partial x}\right|_{(a_0,a_1)}=\left.\frac{\partial v}{\partial y}\right|_{(a_0,a_1)} \qquad \text{and} \qquad \left.\frac{\partial u}{\partial y}\right|_{(a_0,a_1)}=-\left.\frac{\partial v}{\partial x}\right|_{(a_0,a_1)}.$$

It is convenient to employ the usual shorthand for partial derivatives

$$u_x(a_0,a_1)=v_y(a_0,a_1) \qquad \text{and} \qquad u_y(a_0,a_1)=-v_x(a_0,a_1).$$

Now suppose that f is holomorphic on U. Then these equations are valid on the whole of U and so they reduce to

$$u_x=v_y \qquad \text{and} \qquad u_y=-v_x.$$

They are called the **Cauchy-Riemann equations**. We have shown so far that if f is holomorphic then the Cauchy-Riemann equations hold.

In fact the opposite is true:

Theorem 11.1. *Let $f\colon U \longrightarrow \mathbb{C}$ be a continuous function.*

Then f is holomorphic if and only if the partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations.

(11.1) is quite striking, since it says a two dimensional limit exists, provided the limit along a horizontal and vertical line are the same. (11.1) is the strongest known version; it is common to require that the partial derivatives exist and that they are continuous. Due to the significance of (11.1) the Cauchy-Riemann equations are amongst the most famous set of PDEs (partial differential equations). We will prove the converse direction of (11.1) later in the class (with stronger hypotheses on f).

Example 11.2. *Let* $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ *be the function* $f(z) = z^2$.

We have already seen that f is holomorphic so that it is entire. We have

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v(x, y) = 2xy.$$

In this case

$$\begin{aligned} u_x &= 2x & u_y &= -2y \\ v_x &= 2y & v_y &= 2x. \end{aligned}$$

Thus

$$\begin{aligned} u_x &= 2x \\ &= v_y, \end{aligned}$$

and

$$\begin{aligned} u_y &= -2y \\ &= -v_x, \end{aligned}$$

as expected.

Let $f\colon U \longrightarrow \mathbb{C}$ be a function on a region U. Then f is a function of x and y. But given z and $\bar{z}$ we can find x and y,

$$\begin{aligned} x &= \frac{z + \bar{z}}{2} \\ y &= \frac{z - \bar{z}}{2i}. \end{aligned}$$

Thus we may think of f as being a function of z and $\bar{z}$. Informally, a function is holomorphic if and only if it is a function of z, not of $\bar{z}$.