10. Limits and Derivatives

Definition 10.1. *Let $U \subset \mathbb{C}$ be a region and let $f\colon U \longrightarrow \mathbb{C}$ be a function.*

If a is a point of U then we say that $b \in \mathbb{C}$ is the ***limit of f as z approaches*** *a and write*

$$\lim_{z \to a} f(z) = b$$

if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(z) - b| < \epsilon \qquad \text{whenever} \qquad 0 < |z - a| < \delta.$$

Informally, you get closer to b, the closer you get to a. Pictorially, given any disk centred around b, no matter how small, there is always a punctured disk centred around a, so small that we land in the disk centred around b.

As usual we exclude the point a in defining the limit.

Definition 10.2. *Let $U \subset \mathbb{C}$ be a region.*

We say that the function $f\colon U \longrightarrow \mathbb{C}$ is ***continuous at*** *$a \in U$ if the limit of f at a exists and*

$$\lim_{z \to a} f(z) = f(a).$$

Once we have the definition of a limit we can define the derivative in the usual way.

Definition 10.3. *Let $U \subset \mathbb{C}$ be a region and let $f\colon U \longrightarrow \mathbb{C}$ be a function.*

If a is a point of U then we say that f is ***differentiable at a, with derivative*** *$f'(a)$ if*

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = f'(a).$$

Note that it does not make sense to evaluate the ratio when $z = a$, which is one reason we exclude $z = a$ in the definition of the limit. In ordinary calculus the ratio is the slope of a secant line to the curve and in the limit we get the slope of the tangent line. There is no simple geometric meaning behind the ratio.

Up to now, every time we have talked about limits, we could have separated f into its real and imaginary parts and computed two real limits separately. For the derivative of a complex valued function this is a useful thing to do but it is not such a simple story.

Note that limits and therefore derivatives are all computed locally.

Example 10.4. *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a constant function,*

$$z \longrightarrow b.$$

We check that f is differentiable at $a \in \mathbb{C}$. We have

$$\begin{aligned}\lim_{z\to a} \frac{f(z)-f(a)}{z-a} &= \lim_{z\to a} \frac{b-b}{z-a} \\ &= \lim_{z\to a} 0 \\ &= 0.\end{aligned}$$

Thus f is differentiable everywhere and the derivative is zero.

Example 10.5. *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a constant function,*

$$z \longrightarrow z.$$

We check that f is differentiable at $a \in \mathbb{C}$. We have

$$\begin{aligned}\lim_{z\to a} \frac{f(z)-f(a)}{z-a} &= \lim_{z\to a} \frac{z-a}{z-a} \\ &= \lim_{z\to a} 1 \\ &= 1.\end{aligned}$$

Thus f is differentiable everywhere and the derivative is the constant function $f'(z) = 1$.

To go any further we should state some basic properties of derivatives.

Proposition 10.6. *Let $U \subset \mathbb{C}$ be a region and let $a \in U$. Let $f\colon U \longrightarrow \mathbb{C}$ and $g\colon U \longrightarrow \mathbb{C}$ be two function which are both differentiable at a. Let $\alpha \in U$ be a number.*

(1) The function αf is differentiable at a and the derivative is $\alpha f'(a)$.
(2) The function $f+g$ is differentiable at a and the derivative is $f'(a)+g'(a)$.
(3) The function fg is differentiable at a and the derivative is

$$f'(a)g(a) + f(a)g'(a).$$

(4) If $g(a) \neq 0$ then the function f/g is differentiable at a and the derivative is

$$\frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

(5) If $V \subset \mathbb{C}$ is a region, $f(U) \subset V$, $g\colon V \longrightarrow \mathbb{C}$ is a function that is differentiable at $b = f(a)$ then the composition $g \circ f$ is differentiable at a and the derivative is

$$f'(a)g'(f(a)).$$

These formulas are similar to the usual formulas in a first course in calculus and are proved in the usual way. Note that (1) is in fact a consequence of (3), where g is the constant function $z \longrightarrow \alpha$. (4) is a consequence of (3), (5) and the fact that the derivative of

$$z \longrightarrow \frac{1}{z} \qquad \text{is} \qquad z \longrightarrow -\frac{1}{z^2}.$$

Definition 10.7. *We say that a function $f\colon U \longrightarrow \mathbb{C}$ defined on a region U is **holomorphic** if it is differentiable at every point of U.*

*We say that f is **entire** if f is holomorphic and $U = \mathbb{C}$.*

The constant function

$$z \longrightarrow b \qquad \text{and the identity function} \qquad z \longrightarrow z$$

are entire functions. Entire functions play a very special role in complex variable.

Example 10.8. *If n is a natural number then the function $z \longrightarrow z^n$ is entire and the derivative is*

$$z \longrightarrow nz^{n-1}.$$

We proceed by induction on n. We have already know the result when $n = 0$, since the derivative of a constant is zero and we already know the result when $n = 1$. We have

$$\begin{aligned}
(z^{n+1})' &= (z \cdot z^n)' \\
&= z' \cdot z^n + z \cdot (z^n)' \\
&= 1 \cdot z^n + z \cdot nz^{n-1} \\
&= z^n + nz^n \\
&= (n+1)z^n.
\end{aligned}$$

Here we applied Leibniz to get from line 1 to line 2 and induction to get from line 2 to line 3.

Example 10.9. *If*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

is a polynomial then $p(z)$ is entire and the derivative is

$$p'(z) = na_n z^{n-1} + (n-1)a_{n-1} z^{n-1} + \cdots + a_1.$$

Theorem 10.10. *Let $\sum a_n(z-z_0)^n$ be a power series with radius of convergence R.*

Then $\sum a_n(z-z_0)^n$ is holomorphic on the open disk centred at z_0 with radius R and the derivative is the power series

$$\sum n a_n (z-z_0)^{n-1}$$

centred at z_0 with radius of convergence R.

In words, to differentiate a power series, just differentiate term by term, as though it were a polynomial. The key point is that the partial sums are polynomials which converge uniformly to the power series.

Theorem 10.11. *If $f\colon U \longrightarrow \mathbb{C}$ is analytic on the region U then it is holomorphic.*

Proof. We just need to check this locally. Locally an analytic function is given by a power series and so we just apply (10.10). $\square$

Example 10.12. *The exponential function $z \longrightarrow e^z$ is entire and the derivative is the same function.*

Indeed e^z is analytic with radius of convergence ∞ and so it is entire. If you differentiate the power series

$$1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots$$

term by term you get the same power series.

Example 10.13. *The sine function $z \longrightarrow \sin z$ is entire and the derivative is the cosine function $z \longrightarrow \cos z$.*

Indeed $\sin z$ is analytic with radius of convergence ∞ and so it is entire. If you differentiate the power series

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots,$$

term by term you get the power series for $\cos z$.

Example 10.14. *The cosine function $z \longrightarrow \cos z$ is entire and the derivative is the function $z \longrightarrow -\sin z$.*

Indeed $\cos z$ is analytic with radius of convergence ∞ and so it is entire. If you differentiate the power series

$$1 - \frac{z^2}{4!} + \frac{z^4}{4!} + \dots,$$

term by term you get the power series for $-\sin z$.

Example 10.15. *The function $z \longrightarrow \operatorname{Log} z$ on the region $U = \mathbb{C} - (-\infty, 0]$ is holomorphic and the derivative is the function $z \longrightarrow \frac{1}{z}$.*

By definition the $z \longrightarrow \operatorname{Log} z$ is the inverse of the exponential, meaning that

$$e^{\operatorname{Log} z} = z.$$

It is not hard to check that this implies that $z \longrightarrow \operatorname{Log} z$ is holomorphic. If we differentiate both sides of the equality above and apply the chain rule then we get:

$$(\operatorname{Log} z)' e^{\operatorname{Log} z} = 1.$$

It follows that

$$(\operatorname{Log} z)' = \frac{1}{z}.$$

Example 10.16. *The Riemann zeta function $\zeta(s)$ is holomorphic on the region $U = \mathbb{C} - \{1\}$. The derivative is given by*

$$\sum_{n=1}^{\infty} \frac{-\ln n}{n^s} \qquad \textit{for} \qquad \operatorname{Re}(s) > 1.$$

We already stated that the Riemann zeta function is analytic and so it is certainly holomorphic. We have uniform convergence of the series for $\operatorname{Re}(s) > 1$ and so we can differentiate term by term. We have

$$\begin{aligned}
(n^{-s})' &= (e^{-s\ln n})' \\
&= -\ln n e^{-s\ln n} \\
&= -n^{-s} \ln n \\
&= \frac{-\ln n}{n^s}.
\end{aligned}$$

Let $f \colon U \longrightarrow \mathbb{C}$ be a function on a region U. Then f is a function of x and y. But given z and $\bar{z}$ we can find x and y,

$$\begin{aligned}
x &= \frac{z + \bar{z}}{2} \\
y &= \frac{z - \bar{z}}{2i}.
\end{aligned}$$

Thus we may think of f as being a function of z and $\bar{z}$. Informally, a function is holomorphic if and only if it is a function of z, not of $\bar{z}$.