1. Compare and contrast

Complex numbers were first introduced to solve quadratic equations:

$$ax^2 + bx + c = 0 \qquad \text{has solutions} \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac < 0$ the square root is not a real number. But if we introduce a square root of -1,

$$i = \sqrt{-1} \qquad \text{so that} \qquad i^2 = -1$$

then the quadratic formula gives all of the solutions to every quadratic equations.

From this perspective, complex numbers $z = x + iy$ just seem like a convenience. On the other hand, the same can be said of the number 0, negative numbers and fractions and of course these numbers now seem indispensible.

We will see in this course, time and again, that curious looking coincidences in the real setting will make much more sense in the complex setting.

Complex numbers also seem indispensible to describe the "real" world. In fact, quantum mechanics, which describes how electrons and photons behave (and everything else at this length scale and smaller), can only really be expressed in terms of complex numbers. To move in ordinary space, to decide how to get from a to b, photons use complex numbers.

Real numbers are useful for calculus, for taking limits (this doesn't work very well for rational numbers). Similarly one can differentiate and integrate complex functions.

However ordinary calculus has a dirty secret. Most of the time, everything goes wrong. The calculus of complex functions is completely the opposite, everything works like a dream.

For example it is easy to write down a continuous real function which is not differentiable at a point. Start with the function

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

This is called the **Heaviside step function**. H is a function from the real line to the real line:

$$H \colon \mathbb{R} \longrightarrow \mathbb{R}.$$

If you integrate H you get the function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & x > 0. \end{cases}$$

Note that

$$f \colon \mathbb{R} \longrightarrow \mathbb{R}$$

is continuous. The two pieces $x \longrightarrow 0$ and $x \longrightarrow x$ are continuous and they agree at the origin. Integration tends to make functions "smoother ".

If you differentiate f then you get H (the fundamental theorem of calculus). But H is not continuous at the origin and it is clear that f is not differentiable at the origin, even though it is differentiable everywhere else.

One can take this example and make it worse. Suppose you integrate H twice, that is, you integrate f once to get

$$g \colon \mathbb{R} \longrightarrow \mathbb{R}$$

given by

$$g(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & x > 0. \end{cases}$$

Then g is continuous, its derivative is f, which is continuous but the second derivative of g exists everywhere and is continuous, except at the origin, where the second derivative does not exist.

Of course, we can repeat this process ad infinitum, to get

$$f_n \colon \mathbb{R} \longrightarrow \mathbb{R}$$

given by

$$f_n(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^n}{n!} & x > 0. \end{cases}$$

Then the derivative of f_n is f_{n-1}. In particular it follows that $f_0(x) = H(x)$, $f_1(x) = f(x)$ and $f_2(x) = g(x)$. Thus $f_n(x)$ can be differentiated $n - 1$ times but it cannot be differentiated n times.

By contrast, if

$$f \colon \mathbb{C} \longrightarrow \mathbb{C}$$

is a complex function, which is **holomorphic**, meaning f has one derivative which is continuous, then f is infinitely differentiable (sometimes called smooth).

The problem is that the situation for real functions is far worse than this. Recall that many functions in calculus have a power series expansion, sometimes called a Maclaurin or Taylor series. For example,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots.$$

Power series are very useful. One can use them to compute the value of a function with arbitrary precision (and enough patience, or at least access to a good computer). You can write down the derivative or integral of a power series, just do it term by term.

They are the next best thing to polynomials.

The problem is that there are infinitely differentiable real functions whose Taylor series is not equal to the original function.

For example, suppose we start with the function

$$x \longrightarrow e^{-1/x^2}.$$

I claim this gives an infinitely differentiable function

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}.$$

The statement is surely okay outside the origin, since

$$x \longrightarrow \frac{1}{x^2}$$

is an infinitely differentiable function away from the origin,

$$x \longrightarrow e^x$$

is infinitely differentiable everywhere, and the composition of infinitely differentiable functions is infinitely differentiable (using the chain rule).

What happens at the origin? As $x \to 0$, $\frac{1}{x^2} \to \infty$ and so $-\frac{1}{x^2} \to -\infty$. It follows that

$$\lim_{x \to 0} e^{-1/x^2} = 0,$$

so that the limit exists and is equal to zero. It is natural then to define

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

and with this definition we get a continuous function. What happens when we differentiate? Outside the origin we get the function

$$x \longrightarrow \frac{2}{x^3} e^{-1/x^2}$$

If we consider the limit as $x \to 0$ we again get zero (since exponential always beats polynomial). Thus if we define

$$f_1 \colon \mathbb{R} \longrightarrow \mathbb{R}$$

by the rule

$$f_1(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

then $f_1(x)$ is the derivative of $f(x)$. Continuing in this way, it is not hard to see that the function

$$f_n \colon \mathbb{R} \longrightarrow \mathbb{R}$$

defined by the rule

$$f_n(x) = \begin{cases} \frac{p_n(x)}{x^{m_n}} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0, \end{cases}$$

for an appropriate polynomial $p_n(x)$ and an appropriate power x^{m_n} of x, is continuous and it is the derivative of f_{n-1}.

It follows that f is infinitely differentiable. What is the Taylor series of f at zero, that is, what is its Maclaurin series? Since all the derivatives of f are zero at the origin the coefficients of the Taylor series are all zero and the Taylor series of f is the zero power series.

But f is not the zero function. For example,

$$f(1) = \frac{1}{e} \neq 0.$$

In fact

$$\lim_{x \to \pm\infty} f(x) = \infty.$$

We say a function is **analytic** if it has a power series expansion, or what comes to the same thing, if it it is equal to its Taylor series.

Most real functions are not analytic, even when they are infinitely differentiable. By contrast every holomorphic function is analytic, that is, every complex function which has one continuous derivative automatically has a power series expansion.

We will see later that one can integrate holomorphic (or more generally complex) functions along paths (that is, we can compute line integrals). This story is too rich to explain now but let me mention two things. Firstly one can use complex integrals to compute definite integrals of real functions, even those whose integrands are not the derivative of an elementary function.

Secondly, since the paths are in the complex plane, topology starts to play a very interesting role in the whole story.