HOMEWORK 8, DUE FRIDAY MARCH 6TH, 12PM

1. Let f be an entire function. If the real part of f is bounded from above then show that f is constant. (*Hint: consider* $e^{f(z)}$.)
2. Suppose that f is an entire function such that $f(z)/z^n$ is bounded for $|z| \geq R$. Show that f is a polynomial of degree at most n.
3. Expand the following functions in power series about ∞:
(a)
$$\frac{1}{z^2+1};$$
(b)
$$\frac{z^2}{z^3-1};$$
(c)
$$e^{1/z^2};$$
(d)
$$z \sinh(1/z).$$
4. Let E be a closed bounded subset of the complex plane $\mathbb{C}$ over which area can be defined and set
$$f(w) = \iint_E \frac{\mathrm{d}x\mathrm{d}y}{w-z} \qquad \text{where} \qquad w \in U = \mathbb{C} \setminus E$$
and $z = x + iy$. Show that f is holomorphic at ∞ and find a formula for the coefficients.
5. Find the zeroes and their orders of the following functions:
(a)
$$\frac{z^2+1}{z^2-1}.$$
(b)
$$\frac{1}{z} + \frac{1}{z^5}.$$
(c)
$$z^2 \sin z.$$

Challenge Problems: (Just for fun)

6. Consider the power series
$$\sum_n z^{n!} = z + z^2 + z^6 + z^{24} + z^{120} + \dots.$$

Show that the radius of convergence is one so that

$$f(z) = \sum_n z^{n!}$$

is a holomorphic function on the open unit disk. On the other hand show that

$$\lim_{r \to 1} |f(r\omega)| = \infty$$

where ω is any root of unity and r approaches 1 from below.

7. If f is an entire function and f omits all of the values in a non-empty open disk then show that f is constant.