HOMEWORK 7, DUE FRIDAY FEBRUARY 28TH, 12PM

1. Let $h(z)$ be a continuous function on a curve $\gamma\colon [\alpha, \beta] \longrightarrow \mathbb{C}$. Define a function

$$f\colon U \longrightarrow \mathbb{C} \qquad \text{by the rule} \qquad f(z) = \int_\gamma \frac{h(w)}{w-z}\,\mathrm{d}w,$$

where U is the region $\mathbb{C} \setminus \gamma$, the complement of the image of γ.
Show that f is holomorphic on U by using the limit definition of the derivative. What is $f'(z)$?
2. Evaluate the following integrals:
(a)

$$\oint_{|z|=2} \frac{z^n}{z-1}\,\mathrm{d}z,$$

where $n \geq 0$ is an integer.
(b)

$$\oint_{|z|=1} \frac{z^n}{z-2}\,\mathrm{d}z,$$

where $n \geq 0$ is an integer.
(c)

$$\oint_{|z|=1} \frac{\sin z}{z}\,\mathrm{d}z.$$

(d)

$$\oint_{|z|=1} \frac{\cosh z}{z^3}\,\mathrm{d}z.$$

(e)

$$\oint_{|z|=1} \frac{e^z}{z^m}\,\mathrm{d}z,$$

where m is an integer.
(f)

$$\oint_{|z-1-i|=5/4} \frac{\operatorname{Log} z}{(z-1)^2}\,\mathrm{d}z.$$

(g)

$$\oint_{|z|=1} \frac{\mathrm{d}z}{z^2(z^2-4)e^z}.$$

(h)

$$\oint_{|z-1|=2} \frac{\mathrm{d}z}{z^2(z^2-4)e^z}.$$

3. Show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-itx}\,\mathrm{d}x = e^{-t^2/2}$$

for any real number t, by integrating $e^{-z^2/2}$ around the rectangle with vertices $\pm R$ and $\pm R + it$ and letting R go to infinity.

4. Let f be a holomorphic function on a region U and let u be the real part of f, so that u is a real valued function on U.

Let $a \in U$ and suppose that the closed disk of radius ρ centred about a is contained in U. Show that

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + \rho e^{i\theta})\,\mathrm{d}\theta.$$

This is known as the *mean value property of harmonic functions.*

Challenge Problems: (Just for fun)

4. (continued). Show that if u achieves its maximum, or minimum, on U then u is constant.

5. Prove the fundamental theorem of algebra using the following line of proof.

Let $p(z)$ be a polynomial with no zeroes on $\mathbb{C}$. Our goal is to show that p is a constant polynomial.

We may write

$$p(z) = p(0) + zq(z),$$

for some polynomial $q(z)$. Divide this expression by $zp(z)$ and integrate around a large circle.