HOMEWORK 3, DUE FRIDAY JANUARY 31ST, 12PM

1. (a) Show that the function

$$M\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \qquad \text{given by} \qquad M(z) = \frac{2iz + 1 - i}{2z - 1 + i}$$

takes 0 to -1, 1 to 1 and ∞ to i.
(b) Show that M does not take the real line, the line $y = 0$, to a line.
(c) Assuming that M sends lines and circles to lines and circles, show that M takes the real line to the unit circle.
(d) Check this by direction calculation.
2. Write down a Möbius transformation that takes 0 to 1, 1 to $1 + i$ and ∞ to 2.
3. Show that

$$e^{\bar{z}} = \overline{e^z}.$$

4. Find the images of the following regions under the exponential map, $z \longrightarrow e^z$:
(a)

$$\{\, z \in \mathbb{C} \mid 0 < \operatorname{Re}(z) < 1 \,\}.$$

(b)

$$\{\, z \in \mathbb{C} \mid |z| \leq \pi/2 \,\}.$$

(c)

$$\{\, z \in \mathbb{C} \mid |z| \leq \pi \,\}.$$

(d)

$$\{\, z \in \mathbb{C} \mid |z| \leq 3\pi/2 \,\}.$$

5. Calculate
(a) $\operatorname{Log}(2)$
(b) $\operatorname{Log}(i)$
(c) $\operatorname{Log}(1 + i)$
(d) $\operatorname{Log}(1 + i\sqrt{3})/2$.

Challenge Problems: (Just for fun)

6. (a) If

$$M\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \qquad \text{given by} \qquad M(z) = \frac{az + b}{cz + d}$$

is a Möbius transformation then show that the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an invertible 2×2 matrix with complex entries. (The matrix A isn't unique, as we can rescale a, b, c and d; in practice this doesn't matter)
(b) If M and N are two Möbius transformations and A and B are two associated matrices then show that the composition $M \circ N$ of M and N is the Möbius transformations given by the matrix product AB (there isn't really a better way to do this other than direct computation).
(c) Show that every Möbius transformation is invertible, that the inverse is a Möbius transformation, and give a simple formula for this Möbius transformation.