HOMEWORK 2, DUE FRIDAY JANUARY 24TH, 12PM

1. Find formulas for

$$\cos 4\theta \qquad \text{and} \qquad \sin 4\theta,$$

involving only $\cos\theta$ and $\sin\theta$.
2. (a) Show that

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1).$$

(b) Show that if ζ is an nth root of unity then either $\zeta = 1$ or

$$\zeta^{n-1} + \zeta^{n-2} + \cdots + \zeta + 1 = 0.$$

3. Let z and w be complex numbers.
(a) Show that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

(b) Show that cos and sin are periodic functions with period 2π.
(c) Prove the addition formulas:

$$\cos(z+w) = \cos z \cos w - \sin z \sin w$$
$$\sin(z+w) = \cos z \sin w + \sin z \cos w.$$

4. The functions

$$\sinh\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{and} \qquad \cosh\colon \mathbb{C} \longrightarrow \mathbb{C}$$

are defined by

$$\cosh(z) = \cos(iz) \qquad \text{and} \qquad \sinh(z) = -i\sin(iz).$$

(a) Show that

$$\cos z = \cos x \cosh y - i \sin x \sinh y \qquad \text{and} \qquad \sin z = \sin x \cosh y + i \cos x \sinh y$$

(b) Show that

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \qquad \text{and} |\sin z|^2 = \sin^2 x + \sinh^2 y$$

where $z = x + iy$.
(c) Find all zeroes of the cosine and sine functions, that is, find all solutions of

$$\cos z = 0 \qquad \text{and of} \qquad \sin z = 0.$$

(d) Find all periods of the cosine and sine functions.
5. (a) Show that the function $z \longrightarrow z^2$ maps the region

$$\{\, z \in \mathbb{C} \mid \operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0 \,\}$$

that is, the first quadrant, to the region

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \},$$

that is, the upper half plane.
(b) Find a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ that maps the region

$$\{ z \in \mathbb{C} \mid 0 < \operatorname{Arg}(z) < \pi/3 \}$$

to the upper half plane

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}.$$

(c) Find a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ that maps the region

$$\{ z \in \mathbb{C} \mid 0 < \operatorname{Arg}(z) < \pi/n \}$$

to the upper half plane

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}.$$

(d) Show that the function $z \longrightarrow 1/z$, maps the region

$$\{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$$

that is, the punctured unit disc, to the region

$$\{ z \in \mathbb{C} \mid 1 < |z| \}$$

that is, the outside of the unit disc.

Challenge Problems: (Just for fun)

6. Find all possible values of i^i. How about

$$i^{i^i}?$$

7. Find a function that maps the upper half plane

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$$

to the unit disc:

$$\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}.$$

8. Given three distinct points p, q and r of the extended complex plane (so that p, q and r are either complex numbers or ∞) show that there is a unique Möbius transformation

$$z \longrightarrow \frac{az + b}{cz + d}$$

taking p to 0, q to 1 and r to ∞.