HOMEWORK 1, DUE FRIDAY JANUARY 17TH, 12PM

1. Let a be a complex number and let $\rho > 0$ be a positive real. Show that the equation

$$|z|^2 - 2\,\mathrm{Re}(\bar{a}z) + |a|^2 = \rho^2$$

represents a circle centred at a with radius ρ.

2. Consider the polynomial $p(z) = z^3 + z^2 + z + 1$.

(a) Verify that i is a root of $p(z)$.

(b) Find the other roots.

3. For which integers n is i an nth root of unity?

4. Let $n \geq 1$ be an integer.

(a) Show that

$$1 + z + z^2 + z^3 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}.$$

(b) Show that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin(n + \frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

5. Show that

$$\left(\frac{1 + i\tan\theta}{1 - i\tan\theta}\right)^n = \frac{1 + i\tan n\theta}{1 - i\tan n\theta}$$

for every integer n.

6. Find the distinct sixth roots of -64.

7. Show that

$$(1 - \sqrt{3}i)^{10} = 2^9(-1 + \sqrt{3}i).$$

Challenge Problems: (Just for fun)

8. Prove the triangle inequality

$$|z + w| \leq |z| + |w|,$$

with equality if and only if either $z = 0$ or w is a positive real scalar multiple of z.