TAKE HOME MIDTERM EXAM
MATH 120B, UCSD, SPRING 20

You have 24 hours.

There are 6 problems, and the total number of points is 100.

Please make your work as clear and easy to follow as possible. There is no need to be verbose but explain all of the steps, using your own words. You may consult the lecture notes and model answers but you may not use any other reference nor may you confer with anyone. You may use any of the standard results in the lecture notes as long as you clearly state what you are using. If you don't know how to solve the whole problem answer the portion you can solve.

Please submit your answers on Gradescope by 5pm on Thursday April 30th.

1. (3pts) (i) It is wrong.
(ii) It is very unfair to other students taking the class.
(iii) It does not help to learn the material.
2. (17pts) Let $\gamma = \gamma_1 + \gamma_2$ be the standard contour and let

$$f(z) = \frac{1+z}{1+z^3} = \frac{1}{1-z+z^2}.$$

Then $f(z)$ has isolated singularities at the cube roots of -1, apart from -1 itself. Of these only one is in the upper half plane, at $e^{\pi i/3}$. The residue there is

$$\begin{aligned}
\operatorname{Res}_{e^{\pi i/3}} f(z) &= \lim_{z \to e^{\pi i/3}} \frac{z - e^{\pi i/3}}{1-z+z^2} \\
&= \lim_{z \to e^{\pi i/3}} \frac{1}{z - e^{5\pi i/3}} \\
&= \frac{1}{e^{\pi i/3} - e^{5\pi i/3}} \\
&= \frac{1}{2i} \frac{2i}{e^{\pi i/3} - e^{-\pi i/3}} \\
&= \frac{1}{2i} \frac{1}{\sin \pi/3} \\
&= \frac{1}{2i} \frac{2}{\sqrt{3}}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{1+z}{1+z^3}\,\mathrm{d}z &= 2\pi i \operatorname{Res}_{e^{\pi i/3}} f(z) \\
&= \frac{2\pi}{\sqrt{3}}.
\end{aligned}$$

We estimate the integral around γ_2, the semicircle of radius R centred at the origin in the upper half plane. The length of γ_2 is $L = \pi R$. The maximum value M of $f(z)$ is at most

$$\begin{aligned}
|f(z)| &= \frac{1}{|1-z+z^2|} \\
&\leq \frac{1}{R^2 - R - 1}.
\end{aligned}$$

It follows that

$$\left|\int_{\gamma_2} \frac{1+z}{1+z^3}\,\mathrm{d}z\right| \leq LM$$
$$\leq \frac{\pi R}{R^2 - R - 1},$$

which goes to zero as R goes to infinity.
On the other hand

$$\int_{\gamma_1} \frac{1+z}{1+z^3}\,\mathrm{d}z = \int_{-R}^{R} \frac{1+x}{1+x^3}\,\mathrm{d}x.$$

As the original improper integral converges it follows that if we let R go to infinity we get the integral we are trying to compute.
Hence

$$\int_{-\infty}^{\infty} \frac{1+x}{1+x^3}\,\mathrm{d}x = \frac{2\pi}{\sqrt{3}}.$$

3. (20pts) We integrate around the unit circle, using the substitution

$$z = e^{i\theta} \qquad \text{so that} \qquad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}.$$

In this case

$$\cos\theta = \frac{z + 1/z}{2} \qquad \text{and} \qquad \sin\theta = \frac{z - 1/z}{2i}.$$

We have

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{(\cos\theta)^4 + (\sin\theta)^4} &= \oint_{|z|=1} \frac{\mathrm{d}z}{iz((z+1/z)^4/2^4 + (z-1/z)^4/(2i)^4)} \\
&= \frac{2^4}{i} \oint_{|z|=1} \frac{z^3\mathrm{d}z}{(z^2+1)^4 + (z^2-1)^4} \\
&= \frac{2^3}{i} \oint_{|z|=1} \frac{z^3\mathrm{d}z}{z^8 + 6z^4 + 1}.
\end{aligned}$$

Consider the integrand

$$f(z) = \frac{z^3}{z^8 + 6z^4 + 1}.$$

This has singularities at the zeroes of

$$z^8 + 6z^4 + 1 = (z^4 + 3)^2 - 8.$$

We get

$$(z^4 + 3)^2 = 8 \qquad \text{so that} \qquad z^4 = -3 \pm 2\sqrt{2}.$$

Now $|z| \leq 1$ if and only if $|z^4| \leq 1$ and so the only singularities in the unit disk are given by the fourth roots of

$$2\sqrt{2} - 3 \in (-1, 0).$$

These are

$$\alpha e^{\pi i/4}; \qquad \alpha e^{3\pi i/4}; \qquad \alpha e^{5\pi i/4} \qquad \text{and} \qquad \alpha e^{7\pi i/4},$$

where $\alpha \in (0, 1)$ is the real fourth root of $3 - 2\sqrt{2} \in (0, 1)$.
We calculate residues at these points. All poles are simple. We have

$$\begin{aligned}
\operatorname{Res}_{\alpha e^{\pi i/4}} f(z) &= \lim_{z \to \alpha e^{\pi i/4}} \frac{z^3}{8z^7 + 24z^3} \\
&= \frac{1}{8} \lim_{z \to \alpha e^{\pi i/4}} \frac{1}{z^4 + 3} \\
&= \frac{1}{8} \frac{1}{2\sqrt{2} - 3 + 3} \\
&= \frac{1}{16\sqrt{2}}.
\end{aligned}$$

It is clear the other singularities will give the same result. The residue theorem implies that

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{(\cos\theta)^4 + (\sin\theta)^4} &= \frac{8}{i} \oint_{|z|=1} \frac{z^3 dz}{z^8 + 6z^4 + 1} \\
&= 4 \cdot \frac{16\pi}{16\sqrt{2}} \\
&= 2\sqrt{2}\pi.
\end{aligned}$$

4. (20pts) We integrate over

$$f(z) = \frac{z^3 e^{iz}}{(1 + z^2)^2}$$

over the standard contour

$$\gamma = \gamma_1 + \gamma_2,$$

as in Question 2.
We assume that $R > 1$ so that we capture the only isolated singularity in the upper half plane at i. As this is a double pole the residue there

is

$$\begin{aligned}
\operatorname{Res}_i f(z) &= \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^3 e^{iz}}{(z+i)^2}\right) \\
&= \lim_{z\to i} \frac{(3z^2e^{iz} + z^3e^{iz})(z+i)^2 - 2(z+i)z^3e^{iz}}{(z+i)^4} \\
&= \lim_{z\to i} \frac{z^2e^{iz}((3+z)(z+i) - 2z)}{(z+i)^3} \\
&= \frac{-e^{-1}((3+i)(2i) - 2i)}{(2i)^3} \\
&= \frac{-e^{-1}((3+i) - 2)}{(2i)^2} \\
&= \frac{1+i}{4e}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{z^3e^{iz}}{(1+z^2)^2}\,\mathrm{d}z &= 2\pi i \operatorname{Res}_i f(z) \\
&= \pi\frac{i-1}{2e}.
\end{aligned}$$

Next we show the integrals over the semicircle goes to zero as we increase R to infinity. We have

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{z^3e^{iz}}{(1+z^2)^2}\,\mathrm{d}z\right| &\le \int_{\gamma_2} \frac{|z^3e^{iz}|}{|(1+z^2)^2|}\,|\mathrm{d}z| \\
&\le \frac{R^3}{(R^2-1)^2}\int_{\gamma_2} |e^{iz}\,|\mathrm{d}z| \\
&< \frac{\pi R^3}{(R^2-1)^2},
\end{aligned}$$

which goes to zero, as R goes to infinity. To get from line two to line three we applied Jordan's Lemma.
For γ_1 have

$$\int_\gamma \frac{z^3e^{iz}}{(1+z^2)^2}\,\mathrm{d}z = \int_{-R}^{R} \frac{x^3e^{ix}}{(1+x^2)^2}\,\mathrm{d}x.$$

Taking the limit as R approaches ∞ we get

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x^3e^{ix}}{(1+x^2)^2}\,\mathrm{d}x = \pi\frac{i-1}{2e}.$$

Taking the imaginary part of both sides we get

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 \sin x}{(1+x^2)^2} \, \mathrm{d}x = \frac{\pi}{2e}.$$

As the integrand

$$\frac{x^3 \sin x}{(1+x^2)^2}$$

is even, convergence of the Cauchy principal value implies convergence of the improper integral to the same limit. Thus

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(1+x^2)^2} \, \mathrm{d}x = \frac{\pi}{2e}.$$

5. (20pts) Let

$$f(z) = \frac{(\log z)^2}{(z^2+1)^2}.$$

We have to choose a branch of the logarithm. We cut the complex plane along the negative imaginary axis:

$$V = \mathbb{C} \setminus \{\, iy \,|\, y \leq 0 \,\}.$$

We then choose the branch

$$\log z = \ln |z| + i \arg z \qquad \text{where} \qquad \arg z \in (-\pi/2, 3\pi/2).$$

This has a pole at 0 and so we integrate around the indented contour

$$\gamma = \gamma_- + \gamma_0 + \gamma_+ + \gamma_2,$$

where γ_- goes from $-R$ to $-\rho$, γ_0 goes along the semicircle of radius ρ from $-\rho$ to ρ in the upper half plane, γ_+ goes from ρ to R and γ_2 goes back to $-R$ along the semicircle of radius R in the upper half plane. Note that the region

$$U = \{\, z \in \mathbb{C} \,|\, \rho < |z| < R \,\} \cap \mathbb{H},$$

has boundary γ. $f(z)$ has isolated singularities at $\pm i$ which are both double poles but only the singularity at i belongs to U:

$$\begin{aligned}
\operatorname{Res}_i f(z) &= \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(\log z)^2}{(z+i)^2}\right) \\
&= \lim_{z\to i} \frac{2\log z(z+i)^2/z - 2(z+i)(\log z)^2}{(z+i)^4} \\
&= 2\lim_{z\to i} \frac{\log z(z+i) - z(\log z)^2}{z(z+i)^3} \\
&= 2\frac{\log i(i+i) - i(\log i)^2}{i(2i)^3} \\
&= \frac{2\pi i/2 - (\pi i/2)^2}{4i^3} \\
&= \pi\frac{(4-\pi i)}{16i^2} \\
&= \pi\frac{(\pi i - 4)}{16}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{(\log z)^2}{(z^2+1)^2}\,\mathrm{d}z &= 2\pi i \operatorname{Res}_i f(z) \\
&= 2\pi i\pi\frac{(\pi i - 4)}{16} \\
&= -\pi^2\frac{\pi + 4i}{8}.
\end{aligned}$$

Next we show the integrals over γ_2 and γ_0 go to zero. As usual we have to estimate the largest value of $|f(z)|$. Over γ_2 we have

$$\begin{aligned}
|f(z)| &= \frac{|\log z|^2}{|z^2+1|^2} \\
&\le \frac{(\ln R + 2\pi)^2}{(R^2-1)^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{(\log z)^2}{(z^2+1)^2}\,\mathrm{d}z\right| &\le LM \\
&\le \frac{\pi R(\ln R + 2\pi)^2}{(R^2-1)^2},
\end{aligned}$$

which goes to zero as R goes to infinity. Over γ_0 we have

$$\begin{aligned} |f(z)| &= \frac{|\log z|^2}{|z^2+1|} \\ &\leq \frac{(2\pi - \ln\rho)^2}{(1-\rho^2)^2}. \end{aligned}$$

Thus

$$\begin{aligned} \left|\int_{\gamma_2} \frac{(\log z)^2}{z^2+1}\, \mathrm{d}z\right| &\leq LM \\ &\leq \frac{\pi\rho(2\pi - \ln\rho)^2}{(1-\rho^2)^2}, \end{aligned}$$

which goes to zero as ρ goes to zero, since $\rho(\ln\rho)^2$ goes to zero. The integral over γ_+ is equal to

$$\int_{\gamma_+} \frac{(\log z)^2}{(z^2+1)^2}\, \mathrm{d}z = \int_\rho^R \frac{(\ln x)^2}{(x^2+1)^2}\, \mathrm{d}x.$$

Finally, for the integral over γ_- we use the parametrisation

$$z = -x \qquad \text{where} \qquad x \in [\rho, R].$$

In this case

$$\log z = \ln x + \pi i.$$

This traverses γ_- in the wrong direction, so we flip the sign.

$$\begin{aligned} \int_{\gamma_-} \frac{(\log z)^2}{(z^2+1)^2}\, \mathrm{d}z &= \int_\rho^R \frac{(\ln x + \pi i)^2}{(x^2+1)^2}\, \mathrm{d}x \\ &= \int_\rho^R \frac{(\ln x)^2}{(x^2+1)^2}\, \mathrm{d}x + 2\pi i \int_\rho^R \frac{\ln x}{(x^2+1)^2}\, \mathrm{d}x - \pi^2 \int_\rho^R \frac{1}{(x^2+1)^2}\, \mathrm{d}x. \end{aligned}$$

Letting ρ go to zero and R go to infinity we get:

$$2I = -\pi^2\frac{\pi + 4i}{8} - 2\pi i \int_0^\infty \frac{\ln x}{(x^2+1)^2}\, \mathrm{d}x + \pi^2 \int_0^\infty \frac{1}{(x^2+1)^2}\, \mathrm{d}x.$$

Taking the real part gives

$$I = -\frac{\pi^3}{16} + \frac{1}{2}\pi^2 \int_0^\infty \frac{1}{(x^2+1)^2}\, \mathrm{d}x.$$

We have to calculate the last integral. We use the standard contour. It is clear that the integral goes to zero over the semicircle, as R goes to infinity. There is one pole at i, a double pole. We have

$$\begin{aligned}\operatorname{Res}_i \frac{1}{(z^2+1)^2} &= \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{(z+i)^2}\right)\\ &= -\lim_{z\to i}\frac{2}{(z+i)^3}\\ &= -\frac{2}{(i+i)^3}\\ &= \frac{1}{4i}.\end{aligned}$$

Applying the residue theorem and dividing by 2 we get

$$\int_0^\infty \frac{1}{(x^2+1)^2}\,\mathrm{d}x = \frac{\pi}{4}.$$

Thererefore

$$\int_0^\infty \frac{(\ln x)^2}{(x^2+1)^2}\,\mathrm{d}x = \frac{\pi^3}{16}.$$

6. (20pts) Since we want to calculate an integral over the range $[0,\infty)$ and there is no obvious symmetry, we need to integrate over a keyhole contour. Let

$$f(z) = \frac{z^{a-1}}{(z+b)(z+c)}.$$

Note that $f(z)$ has poles at $-b$ and $-c$ on the real line. So we need to put indented contours around $-b$ and $-c$. We need these indented contours for both γ_+ and γ_-,

$$\gamma = \gamma_+ + \gamma_- + \gamma_2 + \gamma_0 + \gamma_b^+ + \gamma_b^- + \gamma_c^+ + \gamma_c^-.$$

We suppose that the circles around 0, $-b$ and $-c$ all have radius ρ. We need to pick a branch of the logarithm to define the power of z. We pick

$$\log z = \ln|z| + i\arg(z) \qquad \text{where} \qquad \arg(z)\in(0,2\pi).$$

$f(z)$ has no poles inside the region we integrate around and so Cauchy's theorem implies that

$$\int_\gamma \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z = 0.$$

We now deal with each term separately. First the integral around γ_2 the circle of radius R centred at the origin. The length L is $2\pi R$ and for the maximum value of $f(z)$ we have

$$\begin{aligned}|f(z)| &= \frac{|z^{a-1}|}{|(z+b)||(z+c)|}\\ &\leq \frac{R^{a-1}}{(R+b)(R+c)}.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\int_{\gamma_2} \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z\right| &\leq LM\\ &\leq \frac{2\pi R^a}{(R+b)(R+c)},\end{aligned}$$

which goes to zero as R goes to infinity. For the integral around γ_0 we do something similar. The length L is $2\pi\rho$ and for the maximum value of $f(z)$ we have

$$\begin{aligned}|f(z)| &= \frac{|z^{a-1}|}{|(z+b)||(z+c)|}\\ &= \frac{\rho^{a-1}}{(b-\rho)(c-\rho)}.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\int_{\gamma_2} \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z\right| &\leq LM\\ &\leq \frac{2\pi \rho^a}{(b-\rho)(c-\rho)},\end{aligned}$$

which goes to zero as ρ goes to zero, since $a > 0$ and the denominator goes to $bc \neq 0$.
For the remaining integrals, the branch of the logarithm becomes important. For the integral above the cut we have

$$\log z = \ln x \qquad \text{so that} \qquad z^{a-1} = x^{a-1}.$$

For the integrals below the cut we have

$$\log z = \ln x + 2\pi i \qquad \text{so that} \qquad z^{a-1} = x^{a-1}e^{2(a-1)\pi i} = x^{a-1}e^{2a\pi i}.$$

For the integral over γ_+ we have

$$\begin{aligned}\int_{\gamma_+} \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z &= \int_{\rho}^{b-\rho} \frac{x^{a-1}}{(x+b)(x+c)}\,\mathrm{d}x\\ &\quad + \int_{b+\rho}^{c-\rho} \frac{x^{a-1}}{(x+b)(x+c)}\,\mathrm{d}x + \int_{c+\rho}^{R} \frac{x^{a-1}}{(x+b)(x+c)}\,\mathrm{d}x.\end{aligned}$$

For the integral over γ_- we have

$$\int_{\gamma_-} \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z = -e^{2a\pi i}\int_{\rho}^{b-\rho} \frac{x^{a-1}}{(x+b)(x+c)}\,\mathrm{d}x$$
$$- e^{2a\pi i}\int_{b+\rho}^{c-\rho} \frac{x^{a-1}}{(x+b)(x+c)}\,\mathrm{d}x - e^{-2a\pi i}\int_{c+\rho}^{R} \frac{x^{a-1}}{(x+b)(x+c)}\,\mathrm{d}x.$$

Let I be the Cauchy principal value of

$$\int_0^{\infty} \frac{x^{a-1}}{(x+b)(x+c)}\,\mathrm{d}x.$$

Thus if we let ρ go to zero and R go to infinity then the integral over γ_+ approaches I and the integral over γ_- approaches $-e^{2a\pi i}I$ (for those that worry about such things: the Cauchy principal value of both singular points, the one at $-b$ and and the one at $-c$ exist independently of each other so that there is in fact no ambiguity of the meaning of the Cauchy principal value).

Finally consider the integrals over the semircircles around $-b$ and $-c$. Since the poles are simple, we can compute the residue and multiply by the appropriate multiple of πi. At $-b$ we have

$$\begin{aligned}\operatorname{Res}_{-b} f(z) &= \lim_{z\to -b} \frac{z^{a-1}}{z+c}\\ &= \frac{(-b)^{a-1}}{c-b}.\end{aligned}$$

Thus

$$\begin{aligned}\lim_{\rho\to 0}\int_{\gamma_b^+} \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z &= -\pi i \operatorname{Res}_{-b} f(z)\\ &= -\pi i\frac{(-b)^{a-1}}{c-b}.\end{aligned}$$

Similarly

$$\begin{aligned}\lim_{\rho\to 0}\int_{\gamma_c^+} \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z &= -\pi i\frac{(-c)^{a-1}}{b-c}\\ \lim_{\rho\to 0}\int_{\gamma_b^-} \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z &= -\pi i e^{2a\pi i}\frac{(-b)^{a-1}}{c-b}\\ \lim_{\rho\to 0}\int_{\gamma_c^-} \frac{z^{a-1}}{(z+b)(z+c)}\,\mathrm{d}z &= -\pi i e^{2a\pi i}\frac{(-c)^{a-1}}{b-c}.\end{aligned}$$

Putting all of this together gives

$$(1-e^{2a\pi i})I = \frac{\pi i}{c-b}\left((-b)^{a-1} + e^{2\pi i a}(-b)^{a-1} - (-c)^{a-1} - e^{2\pi i a}(-c)^{a-1}\right).$$

Thus

$$\frac{e^{-\pi ia}-e^{a\pi i}}{2i}I=\frac{\pi}{2(c-b)}\left(e^{-\pi ia}(-b)^{a-1}+e^{\pi ia}(-b)^{a-1}-e^{-\pi ia}(-c)^{a-1}-e^{\pi ia}(-c)^{a-1}\right).$$

It follows that

$$-\sin\pi aI=\frac{\pi\cos\pi a}{c-b}\left((-b)^{a-1}-(-c)^{a-1}\right).$$

Hence

$$\int_0^\infty\frac{x^{a-1}}{(x+b)(x+c)}\,\mathrm{d}x=\frac{\pi\cot\pi a}{b-c}\left((-b)^{a-1}-(-c)^{a-1}\right).$$

7. (Extra credit: 10pts) Consider integrating

$$f(z)=\frac{1}{p(z)}$$

around a circle of radius R centred at the origin, where R is large. Then $f(z)$ has isolated singularities at the roots of $p(z)$, so that $f(z)$ has finitely many isolated singularities in the whole complex plane. In particular as long as we avoid one of these singularities we can apply the Residue Theorem.
On the one hand, the Residue Theorem implies that

$$\oint_{|z|=R}f(z)\,\mathrm{d}z=2\pi i\sum_a\operatorname{Res}_a f(z).$$

If R is sufficiently large then we capture every isolated singularity and so the sum on the RHS is constant.
Now consider estimating the integral on the LHS from above. The length L of the circle is $2\pi R$. If

$$p(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0,$$

where $a_n\neq 0$ then

$$|p(z)|>\frac{|a_n|}{2}R^n$$

so that

$$\begin{aligned}|f(z)|&=\frac{1}{|p(z)|}\\&\leq\frac{2}{|a_n|R^n}.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\oint_{|z|=R} f(z)\,\mathrm{d}z\right| &\leq LM \\ &\leq \frac{2R}{|a_n|R^n} \\ &\leq \frac{1}{|a_n|R^{n-1}},\end{aligned}$$

which goes to zero, as we are assuming that $n \geq 2$.
The only possibility is that the sum of the residues of $f(z)$ is zero.
8. (Extra credit: 20pts) Note that

$$z^{2m+1} - 1 = (z-1)(1 + z + z^2 + \cdots + z^{2m})$$

In particular the integrand of the improper integral has no singularities along the real axis, so that the improper integral converges.
It is also possible to guess the limit as m goes to infinity. If you fix a real number $x \in (-1,1)$ and let m go to infinity then x^{2m+1} approaches zero. On the other hand if $x \notin [-1,1]$ then $|x^{2m+1}|$ approaches infinity, as m goes to infinity.
Thus the integrand approaches the function

$$g(x) = \begin{cases} 1-x & x \in [-1,1] \\ 0 & x \notin [-1,1]. \end{cases}$$

Pointwise convergence is clear and in fact one can even establish uniform convergence with a little bit of work. The area under the graph of $g(x)$ is two. So we expect the limit to be 2.
Let

$$f(z) = \frac{z-1}{z^{2m+1}-1}.$$

We integrate $f(z)$ around the standard contour,

$$\gamma = \gamma_1 + \gamma_2.$$

$f(z)$ has isolated singularitities at the $(2m+1)$th roots of unity, apart from 1. The ones in the upper half plane are located at

$$a_k = e^{2\pi i k/(2m+1)} \qquad \text{where} \qquad 1 \leq k \leq m.$$

The residue there is

$$\begin{aligned}
\operatorname{Res}_{a_k} f(z) &= \lim_{z\to a_k} \frac{z-1}{(2m+1)z^{2m}} \\
&= \frac{a_k-1}{(2m+1)a_k^{2m}} \\
&= \frac{a_k(a_k-1)}{2m+1}.
\end{aligned}$$

The sum of the residues is

$$\begin{aligned}
\frac{1}{2m+1}\sum_{k=1}^{m} a_k(a_k-1) &= \frac{1}{2m+1}\sum_{k=1}^{m} a_k^2 - a_k \\
&= \frac{1}{2m+1}\left(\frac{a_1^2-a_{m+1}^2}{1-a_1^2} - \frac{a_1-a_{m+1}}{1-a_1}\right) \\
&= \frac{1}{2m+1}\left(\frac{a_1^2-a_{2m+2}}{1-a_1^2} - \frac{(a_1-a_{m+1})(1+a_1)}{1-a_1^2}\right) \\
&= \frac{1}{2m+1}\left(\frac{a_{m+1}+a_{m+2}-2a_1}{1-a_2}\right) \\
&= \frac{1}{2m+1}\left(\frac{a_m+a_{m+1}-2}{a_{-1}-a_1}\right) \\
&= \frac{1}{2m+1}\left(\frac{(a_{m/2}-a_{-m/2})^2}{a_{-1}-a_1}\right) \\
&= \frac{1}{2m+1}\left(\frac{(2i)^2\sin^2\frac{m\pi}{2m+1}}{(-2i)\sin\frac{2\pi}{2m+1}}\right) \\
&= -\frac{2i}{2m+1}\left(\frac{\cos^2\frac{\pi}{2(2m+1)}}{\sin\frac{2\pi}{2m+1}}\right) \\
&= -\frac{2i}{2m+1}\left(\frac{\cos^2\alpha}{\sin 4\alpha}\right),
\end{aligned}$$

where

$$\alpha = \frac{\pi}{2(2m+1)}.$$

Thus the residue theorem implies that

$$\int_\gamma \frac{z-1}{z^{2m+1}-1}\,\mathrm{d}z = \frac{4\pi}{2m+1}\left(\frac{\cos^2\alpha}{\sin 4\alpha}\right).$$

We estimate the integral over γ_2. The length if πR and the maximum value is at most

$$\begin{aligned} |f(z)| &= \left| \frac{z-1}{z^{2m+1}-1} \right| \\ &= \frac{|z-1|}{|z^{2m+1}-1|} \\ &\leq \frac{R+1}{R^{2m+1}-1}. \end{aligned}$$

Thus

$$\begin{aligned} \left| \int_{\gamma_2} \frac{z-1}{z^{2m+1}-1} \, \mathrm{d}z \right| &\leq LM \\ &\leq \frac{\pi R(R+1)}{R^{2m+1}-1}, \end{aligned}$$

which goes to zero, as R goes to infinity.
For the integral over γ_1 we have

$$\int_{\gamma_2} \frac{z-1}{z^{2m+1}-1} \, \mathrm{d}z = \int_{-R}^{R} \frac{1}{1+x+x^2+\cdots+x^{2m}} \, \mathrm{d}x.$$

Letting R goes infinity we therefore get

$$\int_{-\infty}^{\infty} \frac{1}{1+x+x^2+\cdots+x^{2m}} \, \mathrm{d}x = 8\alpha \left(\frac{\cos^2 \alpha}{\sin 4\alpha} \right).$$

where

$$\alpha = \frac{\pi}{2(2m+1)}.$$

As m goes to infinity, α goes to zero and so

$$\frac{4\alpha}{\sin 4\alpha}$$

approaches 1. As $2\cos^2 \alpha$ approaches 2, the integral approaches 2.