MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. We have
(a) If

$$y = \arctan x \qquad \text{then} \qquad x = \tan y \qquad \text{so that} \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = \sec^2 y = 1+\tan^2 y$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1+\tan^2 y}\\ &= \frac{1}{1+x^2}.\end{aligned}$$

We have

$$\begin{aligned}\frac{\partial}{\partial\psi}\left(2\arctan\left(\frac{1+r}{1-r}\tan\frac{\psi}{2}\right)\right) &= 2\frac{\partial}{\partial\psi}\left(\frac{1+r}{1-r}\tan\frac{\psi}{2}\right)\frac{1}{1+\left(\frac{1+r}{1-r}\tan\frac{\psi}{2}\right)^2}\\ &= \frac{\frac{1+r}{1-r}\sec^2\frac{\psi}{2}}{1+\left(\frac{1+r}{1-r}\tan\frac{\psi}{2}\right)^2}\\ &= \frac{1-r^2}{(1-r)^2\cos^2\frac{\psi}{2}+(1+r)^2\sin^2\frac{\psi}{2}}\\ &= \frac{1-r^2}{1+2r(\sin^2\frac{\psi}{2}-\cos^2\frac{\psi}{2})+r^2}\\ &= \frac{1-r^2}{1-2r\cos\psi+r^2}\\ &= P_r(\psi).\end{aligned}$$

(b)Note that

$$\tan\frac{2\pi-\theta}{2} = -\tan\frac{\theta}{2} \qquad \text{and} \qquad \tan\frac{\pi-\theta}{2} = \cot\frac{\theta}{2}.$$

Note also that

$$\begin{aligned}\tan\alpha+\cot\alpha &= \frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}\\ &= \frac{\cos^2\alpha+\sin^2\alpha}{\cos\alpha\sin\alpha}\\ &= \frac{2}{\sin 2\alpha}.\end{aligned}$$

Suppose that $x = \arctan a$ and $y = \arctan b$ so that $a = \tan x$ and $b = \tan y$. We have

$$\begin{aligned}\tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y} \\ &= \frac{a-b}{1+ab}.\end{aligned}$$

It follows that

$$\arctan a - \arctan b = \arctan\left(\frac{a-b}{1+ab}\right).$$

If we put

$$a = \frac{1+r}{1-r}\tan\frac{2\pi-\theta}{2} \qquad \text{and} \qquad b = \frac{1+r}{1-r}\tan\frac{\pi-\theta}{2}$$

then

$$\begin{aligned}a-b &= \frac{1+r}{1-r}\tan\frac{2\pi-\theta}{2} - \frac{1+r}{1-r}\tan\frac{\pi-\theta}{2} \\ &= -\frac{1+r}{1-r}\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right).\end{aligned}$$

and

$$\begin{aligned}ab &= \frac{1+r}{1-r}\tan\frac{2\pi-\theta}{2}\frac{1+r}{1-r}\tan\frac{\pi-\theta}{2} \\ &= -\frac{(1+r)^2}{(1-r)^2}.\end{aligned}$$

It follows that

$$\begin{aligned}\tan(x-y) &= \frac{a-b}{1+ab} \\ &= \frac{(1+r)(1-r)\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right)}{(1+r)^2 - (1-r^2)} \\ &= \frac{1-r^2}{2r\sin\theta}.\end{aligned}$$

Thus

$$\arctan\left(\frac{1+r}{1-r}\tan\frac{2\pi-\theta}{2}\right) - \arctan\left(\frac{1+r}{1-r}\tan\frac{\pi-\theta}{2}\right) = \arctan\left(\frac{1-r^2}{2r\sin\theta}\right).$$

2. Let

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\frac{e^{i\phi}+z}{e^{i\phi}-z}\,\mathrm{d}\phi + iv(0).$$

Then $g(z)$ is holomorphic on the unit disk, as it is the line integral of a continuous function.
Consider taking the real part of both sides of the equation above for $g(z)$. If $w = e^{i\phi}$ then (19.2.b) impies that

$$\begin{aligned} P_r(\phi - \theta) &= \operatorname{Re}\left(\frac{w+z}{w-z}\right) \\ &= \operatorname{Re}\left(\frac{e^{i\phi}+z}{e^{i\phi}-z}\right). \end{aligned}$$

It follows that

$$\begin{aligned} \operatorname{Re} g(z) &= \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\frac{e^{i\phi}+z}{e^{i\phi}-z}\,\mathrm{d}\phi + iv(0)\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\operatorname{Re}\left(\frac{e^{i\phi}+z}{e^{i\phi}-z}\right)\mathrm{d}\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})P_r(\phi-\theta)\,\mathrm{d}\phi \\ &= u(z). \end{aligned}$$

As both $v(z)$ and $\operatorname{Im} g(z)$ are harmonic conjugates of $u(z)$, it follows that the difference is a constant. But

$$\begin{aligned} \operatorname{Im} g(0) &= \operatorname{Im}\left(\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\frac{e^{i\phi}}{e^{i\phi}}\,\mathrm{d}\phi + iv(0)\right) \\ &= \operatorname{Im}\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\phi})\,\mathrm{d}\phi + v(0) \\ &= v(0). \end{aligned}$$

Thus $\operatorname{Im} g(0) = v(z)$ and so $f(z) = g(z)$.
3. (a) Note that

$$\tan\frac{\pi/2-\theta}{2} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}.$$

We have

$$
\begin{aligned}
\pi u(r,\theta) &= \int_0^{2\pi} P_r(\phi-\theta)h(e^{i\phi})\,\mathrm{d}\phi \\
&= \int_0^{\pi/2} P_r(\phi-\theta)\,\mathrm{d}\phi \\
&= \arctan\left(\frac{1+r}{1-r}\tan\frac{\pi/2-\theta}{2}\right) - \arctan\left(\frac{1+r}{1-r}\tan\frac{-\theta}{2}\right) \\
&= \arctan\left(\frac{(1-r^2)\left(\tan\frac{\theta}{2}+\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)}{(1-r)^2-(1+r)^2\tan\frac{\theta}{2}\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}}\right) \\
&= \arctan\left(\frac{(1-r^2)\left(\tan\frac{\theta}{2}(1+\tan\frac{\theta}{2})+1-\tan\frac{\theta}{2}\right)}{(1-r)^2(1+\tan\frac{\theta}{2})-(1+r)^2\tan\frac{\theta}{2}(1-\tan\frac{\theta}{2})}\right) \\
&= \arctan\left(\frac{(1-r^2)\left(\tan^2\frac{\theta}{2}+1\right)}{(r^2+1)(\tan^2\frac{\theta}{2}+1)-4r\tan\frac{\theta}{2}+2r(\tan^2\frac{\theta}{2}-1)}\right) \\
&= \arctan\left(\frac{(1-r^2)\sec^2\frac{\theta}{2}}{(r^2+1)\sec^2\frac{\theta}{2}-4r\tan\frac{\theta}{2}+2r(\tan^2\frac{\theta}{2}-1)}\right) \\
&= \arctan\left(\frac{1-r^2}{r^2+1-4r\cos\frac{\theta}{2}\sin\frac{\theta}{2}-2r(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2})}\right) \\
&= \arctan\left(\frac{1-r^2}{r^2+1-2r\sin\theta-2r\cos\theta}\right) \\
&= \arctan\left(\frac{1-x^2-y^2}{x^2+y^2-2y-2x+1}\right) \\
&= \arctan\left(\frac{1-x^2-y^2}{(x-1)^2+(y-1)^2-1}\right).
\end{aligned}
$$

(b) Notice that the

$$\beta\colon \Delta \longrightarrow \mathbb{H} \qquad \text{given by} \qquad \beta(z) = (1-i)\frac{z-i}{z-1}$$

sends to 1 to ∞, i to 0 and -1 to 1.

This gives us the problem of finding a harmonic function on the upper half plane with boundary conditions 0 if the real part is positive and 1 if the real part is negative. We already saw that this has solution

$$\frac{1}{\pi}\arctan\left(\frac{v}{u}\right),$$

where $w = u + iv$.

Now

$$
\begin{aligned}
w &= (1-i)\frac{z-i}{z-1} \\
&= (1-i)\frac{(z-i)(\bar{z}-1)}{|z-1|^2} \\
&= (1-i)\frac{i+|z|^2-z-i\bar{z}}{|z-1|^2} \\
&= \frac{1+i+|z|^2-i|z|^2-z+iz-\bar{z}-i\bar{z}}{|z-1|^2} \\
&= \frac{1+i+|z|^2-i|z|^2-2\operatorname{Re}(z)-2\operatorname{Im}(z)}{|z-1|^2} \\
&= \frac{1+x^2+y^2-2x-2y+i(1-x^2-y^2)}{2x^2+2(y-1)^2} \\
&= \frac{(x-1)^2+(y-1)^2-1+i(-1+x^2+y^2)}{2x^2+2(y+1)^2} \\
&= u+iv.
\end{aligned}
$$

This gives the solution

$$
\frac{1}{\pi}\arctan\left(\frac{1-x^2-y^2}{(x-1)^2+(y-1)^2-1}\right) \qquad \text{where} \qquad \arctan t \in [0,\pi]
$$

4. It is tempting to try to be clever and to adapt the solution from (3) to handle the general case. This consists of simply writing down a biholomorphic map from the unit disk to itself that sends i to $e^{2i\theta_0}$ and fixes 1. This seems like a sensible strategy but the actual computation seems to be quite messy.
Instead is it surprisingly straightforward to compute the Poisson integral formula.
Note that

$$
\tan\frac{2\theta_0-\theta}{2} = \frac{y_0-\tan\frac{\theta}{2}}{1+y_0\tan\frac{\theta}{2}},
$$

where $y_0 = \tan\theta_0$.

We have

$$
\begin{aligned}
\pi u(r,\theta) &= \int_0^{2\pi} P_r(\phi-\theta)h(e^{i\phi})\,\mathrm{d}\phi \\
&= \int_0^{2\theta_0} P_r(\phi-\theta)\,\mathrm{d}\phi \\
&= \arctan\left(\frac{1+r}{1-r}\tan\frac{2\theta_0-\theta}{2}\right) - \arctan\left(\frac{1+r}{1-r}\tan\frac{-\theta}{2}\right) \\
&= \arctan\left(\frac{(1-r^2)\left(\tan\frac{\theta}{2}+\frac{y_0-\tan\frac{\theta}{2}}{1+y_0\tan\frac{\theta}{2}}\right)}{(1-r)^2-(1+r)^2\tan\frac{\theta}{2}\frac{y_0-\tan\frac{\theta}{2}}{1+y_0\tan\frac{\theta}{2}}}\right) \\
&= \arctan\left(\frac{(1-r^2)\left(\tan\frac{\theta}{2}(1+y_0\tan\frac{\theta}{2})+y_0-\tan\frac{\theta}{2}\right)}{(1-r)^2(1+y_0\tan\frac{\theta}{2})-(1+r)^2\tan\frac{\theta}{2}(y_0-\tan\frac{\theta}{2})}\right) \\
&= \arctan\left(\frac{(1-r^2)\left(\tan^2\frac{\theta}{2}+1\right)y_0}{(r^2+1)(\tan^2\frac{\theta}{2}+1)-4ry_0\tan\frac{\theta}{2}+2r(\tan^2\frac{\theta}{2}-1)}\right) \\
&= \arctan\left(\frac{(1-r^2)\sec^2\frac{\theta}{2}y_0}{(r^2+1)\sec^2\frac{\theta}{2}-4ry_0\tan\frac{\theta}{2}+2r(\tan^2\frac{\theta}{2}-1)}\right) \\
&= \arctan\left(\frac{(1-r^2)y_0}{r^2+1-4ry_0\cos\frac{\theta}{2}\sin\frac{\theta}{2}-2r(\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2})}\right) \\
&= \arctan\left(\frac{(1-r^2)y_0}{r^2+1-2ry_0\sin\theta-2r\cos\theta}\right) \\
&= \arctan\left(\frac{(1-x^2-y^2)y_0}{x^2+y^2-2y_0y-2x+1}\right) \\
&= \arctan\left(\frac{(1-x^2-y^2)y_0}{(x-1)^2+(y-y_0)^2-y_0^2}\right).
\end{aligned}
$$

Consider the ratio

$$
\frac{(1-x^2-y^2)y_0}{(x-1)^2+(y-y_0)^2-y_0^2}.
$$

The numerator is zero on the unit circle and positive on the unit disk. On the other hand the denominator is zero on the circle of radius y_0 centred at $1+y_0i$ and it is positive on the open disk of radius y_0 centred at $1+y_0i$. The two circles intersect at 1 and the point $e^{2i\theta_0}$. Thus the ratio is positive and close to zero when r is close to one and $\theta \in (0, 2\theta_0)$ and the ratio is negative and close to zero when r is close to one and $\theta \in (2\theta_0, 2\pi)$.

It follow that $u(r, \theta)$ has the correct behaviour at the boundary.
5. If we let

$$I(\psi) = \int_0^{\psi} P_r(\phi - \theta)\, \mathrm{d}\phi$$

then

$$\frac{\partial I}{\partial \psi} = P_r(\psi - \theta).$$

We have

$$\begin{aligned}\int_0^{2\pi} P_r(\phi - \theta)\delta_h(\phi - \theta_0)\, \mathrm{d}\phi &= 1/h \int_{\theta_0}^{\theta_0 + h} P_r(\phi - \theta)\, \mathrm{d}\phi \\ &= \frac{I(\theta_0 + h) - I(\theta_0)}{h} \\ &= P_r(c - \theta).\end{aligned}$$

for some $c \in (\theta_0, \theta_0 + h)$, where we applied the mean value theorem to get from line two to line three.
We have

$$\begin{aligned}\lim_{h \to 0^+} \int_0^{2\pi} P_r(\phi - \theta)\delta_h(\phi - \theta_0)\, \mathrm{d}\phi &= \lim_{h \to 0^+} P_r(c - \theta) \\ &= P_r(\theta_0 - \theta) \\ &= P_r(\theta - \theta_0).\end{aligned}$$

6. (a) Note that

$$(ae^{i\theta})^n = a^n e^{ni\theta}.$$

It follows that

$$\sum_{n=0}^{\infty} a^n e^{ni\theta} = \frac{1}{1 - ae^{i\theta}}.$$

Taking the real part of both sides gives

$$
\begin{aligned}
1 + 2\sum_{n=0}^{\infty} a^n \cos n\theta &= 2\sum_{n=0}^{\infty} a^n \cos n\theta - 1 \\
&= 2\,\mathrm{Re}\left(\frac{1}{1-ae^{i\theta}}\right) - 1 \\
&= \left(\frac{1}{1-ae^{i\theta}} + \frac{1}{1-ae^{-i\theta}}\right) - 1 \\
&= \frac{1-ae^{-i\theta}+1-ae^{i\theta}}{(1-ae^{i\theta})(1-ae^{-i\theta})} - 1 \\
&= \frac{2-2a\cos\theta}{1-2a\cos\theta+a^2} - 1 \\
&= \frac{1-a^2}{1-2a\cos\theta+a^2}.
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
P_r(\psi) &= \frac{1-r^2}{1-2r\cos\theta+r^2} \\
&= 1 + 2\sum_{n=1}^{\infty} r^n \cos n\psi.
\end{aligned}
$$

(c) Note that the series in (a) is given by a geometric series so that we have uniform convergence and we can compute the Poisson integral formula term by term

$$
\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} P_r(\phi-\theta)h(e^{i\phi})\,\mathrm{d}\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} (1 + 2\sum_{n=1}^{\infty} r^n \cos n(\phi-\theta))h(e^{i\phi})\,\mathrm{d}\phi \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} 2r^n \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty} \cos n(\phi-\theta))h(e^{i\phi})\,\mathrm{d}\phi \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} 2r^n \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty}(\cos n\phi\cos n\theta + \sin n\phi \sin n\theta)h(e^{i\phi})\,\mathrm{d}\phi \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta),
\end{aligned}
$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} h(e^{i\phi}) \cos n\phi \, \mathrm{d}\phi$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} h(e^{i\phi}) \sin n\phi \, \mathrm{d}\phi.$$

Challenge Problems: (Just for fun)