MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. We have to compute the sum of the residues of $e^{st}F(s)$.
(a) Note that

$$F(s) = \frac{2s^3}{s^4-4}$$

has simple poles at $\pm\sqrt{2}$ and $\pm\sqrt{2}i$. Thus $e^{st}F(s)$ also has simple poles at $\pm\sqrt{2}$ and $\pm\sqrt{2}i$. We have

$$\begin{aligned}\operatorname{Res}_{\sqrt{2}i}\frac{2s^3e^{st}}{s^4-4} &= \lim_{s\to\sqrt{2}i}\frac{2s^3e^{st}}{4s^3}\\ &= \lim_{s\to\sqrt{2}i}\frac{e^{st}}{2}\\ &= \frac{e^{\sqrt{2}it}}{2}.\end{aligned}$$

Similarly

$$\operatorname{Res}_{-\sqrt{2}i}\frac{2s^3e^{st}}{s^4-4} = \frac{e^{-\sqrt{2}it}}{2} \qquad \operatorname{Res}_{\sqrt{2}}\frac{2s^3e^{st}}{s^4-4} = \frac{e^{\sqrt{2}t}}{2} \qquad \text{and} \qquad \operatorname{Res}_{-\sqrt{2}}\frac{2s^3e^{st}}{s^4-4} = \frac{e^{-\sqrt{2}t}}{2}.$$

Thus

$$\begin{aligned}f(t) &= \operatorname{Res}_{\sqrt{2}i}\frac{2s^3e^{st}}{s^4-4} + \operatorname{Res}_{-\sqrt{2}i}\frac{2s^3e^{st}}{s^4-4} + \operatorname{Res}_{\sqrt{2}}\frac{2s^3e^{st}}{s^4-4} + \operatorname{Res}_{-\sqrt{2}}\frac{2s^3e^{st}}{s^4-4}\\ &= \frac{e^{\sqrt{2}it}}{2} + \frac{e^{-\sqrt{2}it}}{2} + \frac{e^{\sqrt{2}t}}{2} + \frac{e^{-\sqrt{2}t}}{2}\\ &= \cos\sqrt{2}t + \cosh\sqrt{2}t.\end{aligned}$$

(b) Note that

$$F(s) = \frac{2s-2}{(s+1)(s^2-2s+5)}.$$

has simple poles at -1 and $1\pm 2i$. Thus $e^{st}F(s)$ also has simple poles at -1 and $1\pm 2i$. We have

$$\begin{aligned}\operatorname{Res}_{-1}\frac{(2s-2)e^{st}}{(s+1)(s^2-2s+5)} &= \lim_{s\to-1}\frac{(2s-2)e^{st}}{s^2-2s+5}\\ &= \frac{-4e^{-t}}{8}\\ &= -\frac{e^{-t}}{2}.\end{aligned}$$

We also have

$$\begin{aligned}
\operatorname{Res}_{1+2i} \frac{(2s-2)e^{st}}{(s+1)(s^2-2s+5)} &= \lim_{s\to 1+2i} \frac{(2s-2)e^{st}}{(s+1)(2s-2)} \\
&= \lim_{s\to 1+2i} \frac{e^{st}}{s+1} \\
&= \frac{e^{(1+2i)t}}{2(1+i)} \\
&= (1-i)\frac{e^{(1+2i)t}}{4}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\operatorname{Res}_{1-2i} \frac{(2s-2)e^{st}}{(s+1)(s^2-2s+5)} &= \frac{e^{(1-2i)t}}{2(1-i)} \\
&= (1+i)\frac{e^{(1-2i)t}}{4}.
\end{aligned}$$

Thus

$$\begin{aligned}
f(t) &= \operatorname{Res}_{-1} F(s)e^{st} + \operatorname{Res}_{1+2i} F(s)e^{st} + \operatorname{Res}_{1-2i} F(s)e^{st} \\
&= -\frac{e^{-t}}{2} + (1-i)\frac{e^{(1+2i)t}}{4} + (1+i)\frac{e^{(1-2i)t}}{4} \\
&= -\frac{e^{-t}}{2} + e^t(1-i)\frac{e^{2it}}{4} + (1+i)e^t\frac{e^{-2it}}{4} \\
&= -\frac{e^{-t}}{2} + \frac{1}{2}e^t\cos 2t + \frac{1}{2}e^t \sin 2t.
\end{aligned}$$

(c) Note that

$$F(s) = \frac{12}{s^3+8}$$

has simple poles at

$$2e^{\pi i/3} \qquad -2 \qquad \text{and} \qquad 2e^{5\pi i/3}.$$

Thus $e^{st}F(s)$ also has simple poles at the same points. We have

$$\begin{aligned}
\operatorname{Res}_{2e^{\pi i/3}} \frac{12e^{st}}{s^3+8} &= \lim_{s\to 2e^{\pi i/3}} \frac{12e^{st}}{3s^2} \\
&= \frac{12e^{2te^{\pi i/3}}}{12e^{2\pi i/3}} \\
&= e^{2te^{\pi i/3}}e^{4\pi i/3} \\
&= e^{t+i\sqrt{3}t}e^{4\pi i/3}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\operatorname{Res}_{2e^{5\pi i/3}} \frac{12e^{st}}{s^3+8} &= \lim_{s\to 2e^{5\pi i/3}} \frac{12e^{st}}{3s^2} \\
&= \frac{12e^{2te^{5\pi i/3}}}{12e^{4\pi i/3}} \\
&= e^{2te^{5\pi i/3}} e^{2\pi i/3} \\
&= e^{t-i\sqrt{3}t} e^{2\pi i/3}.
\end{aligned}$$

We also have

$$\begin{aligned}
\operatorname{Res}_{-2} \frac{12e^{st}}{s^3+8} &= \lim_{s\to -2} \frac{12e^{st}}{3s^2} \\
&= \frac{12e^{-2t}}{12} \\
&= e^{-2t}.
\end{aligned}$$

Thus

$$\begin{aligned}
f(t) &= \operatorname{Res}_{-2} \frac{12e^{st}}{s^3+8} + \operatorname{Res}_{2e^{\pi i/6}} \frac{12e^{st}}{s^3+8} + \operatorname{Res}_{2e^{5\pi i/6}} \frac{12e^{st}}{s^3+8} \\
&= e^{-2t} + e^{t+i\sqrt{3}t} e^{4\pi i/3} + e^{t-i\sqrt{3}t} e^{2\pi i/3} \\
&= e^{-2t} + e^{t}\left(e^{it\sqrt{3}} e^{4\pi i/3} + e^{-i\sqrt{3}t} e^{2\pi i/3}\right) \\
&= e^{-2t} + e^{t}(-\cos\sqrt{3}t + \sqrt{3}\sin\sqrt{3}t).
\end{aligned}$$

2. It is easy to see that $z \longrightarrow az+b$ is a biholomorphic map, with inverse

$$z \longrightarrow \frac{z-b}{a}.$$

Conversely, let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a biholomorphic map. Consider the behaviour of f at infinity. As f is entire and not constant it must be unbounded as it approaches infinity.

In particular f must have a singularity at infinity. Suppose that the singularity is essential. The Casorati-Weierstrass theorem implies that f approaches every single complex number $a \in \mathbb{C}$. This is impossible as f is a bijection.

Thus f has a pole at infinity. It follows that f is a rational function

$$f(z) = \frac{p(z)}{q(z)},$$

where $p(z)$ and $q(z)$ are polynomials. If $p(z)$ and $q(z)$ have a common zero then they share the same linear factor. Cancelling, we may assume that $p(z)$ and $q(z)$ have no common zeroes.

Suppose that $q(z)$ has positive degree. Then $q(z)$ must have a zero and this would be a pole of $f(z)$, which is not possible, as f is entire. Thus $f(z)$ is a polynomial.
If the degree of $f(z)$ is at least two, then the derivative of $f(z)$ is a polynomial of degree at least one. But then the derivative has a zero, which is impossible as f is biholomorphic.
Thus $f(z)$ is a polynomial of degree at most one, so that

$$f(z) = az + b,$$

where a and $b \in \mathbb{C}$. $a \neq 0$, otherwise $f(z)$ is constant.
3. We have already seen that Möbius transformations give biholomorphic maps of the extended complex plane.
Conversely, let f be a biholomorphic map of the extended complex plane. Suppose that f sends ∞ to a. If $a = \infty$ then let

$$\alpha \colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

be the identity. Otherwise let α be the biholomorphic map of the extended complex plane given by

$$z \longrightarrow \frac{1}{z-a}$$

Then $g = \alpha \circ f$ is a biholomorphic map of the extended complex plane that sends ∞ to ∞. By what we already proved $g(z) = az + b$. In particular g is a Möbius transformation. Thus the inverse of g is a Möbius transformation and so f is a Möbius transformation.
4. Suppose that

$$f \colon \Delta \longrightarrow \Delta$$

is a biholomorphic map with fixed point a. Let

$$\alpha \colon \Delta \longrightarrow \Delta$$

be the biholomorphic map

$$\alpha(z) = \frac{z-a}{1-\bar{a}z}$$

so that $\alpha(a) = 0$. Let β be the inverse of α. Then

$$g = \alpha \circ f \circ \beta \colon \Delta \longrightarrow \Delta$$

is a biholomorphic map that fixes zero. It follows that g is a rotation. In particular if g has more than one fixed point it is the identity.
Note that b is a fixed point of f if and only if $c = \alpha(b)$ is a fixed point of g. Thus if f has more than one fixed point then g has more than one fixed point and so g is the identity. But then f is the identity.

5. We first do long division to find $p_\infty(z)$. Note that

$$z^6 = (z^2+2z+2)[(z^2+1)(z-1)^2] + [2z^3 - z^2 + 2z - 2].$$

It follows that $p_\infty(z) = z^2 + 2z + 2$ and

$$\frac{z^6}{(z^2+1)(z-1)^2} = z^2 + 2z + 2 + \frac{2z^3 - z^2 + 2z - 2}{(z^2+1)(z-1)^2}.$$

Now

$$\frac{2z^3 - z^2 + 2z - 2}{(z^2+1)(z-1)^2}$$

has poles at $\pm i$ and 1. The poles at $\pm i$ are simple but 1 is a double pole. It follows that

$$\frac{2z^3 - z^2 + 2z - 2}{(z^2+1)(z-1)^2} = \frac{\alpha}{z-i} + \frac{\beta}{z+i} + \frac{\gamma}{z-1} + \frac{\delta}{(z-1)^2},$$

where the first term is the principal part at i, the second term is the principal part at $-i$, the last two terms are the principal part at 1, and α, β, γ and δ are to be determined.
The first three coefficients are residues. We have

$$\begin{aligned}\alpha &= \operatorname{Res}_i \frac{2z^3 - z^2 + 2z - 2}{(z^2+1)(z-1)^2}\\ &= \lim_{z\to i} \frac{2z^3 - z^2 + 2z - 2}{(z+i)(z-1)^2}\\ &= \frac{-1}{2i(i-1)^2}\\ &= \frac{i(i+1)^2}{8}\\ &= -\frac{1}{4}.\end{aligned}$$

Taking complex conjugates we see that

$$\beta = -\frac{1}{4}.$$

At 1 we have

$$\begin{aligned}\alpha &= \operatorname{Res}_1 \frac{2z^3 - z^2 + 2z - 2}{(z^2+1)(z-1)^2} \\ &= \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{2z^3 - z^2 + 2z - 2}{z^2+1}\right) \\ &= \lim_{z\to 1} \frac{(6z^2 - 2z + 2)(z^2+1) - 2z(2z^3 - z^2 + 2z - 2)}{(z^2+1)^2} \\ &= \frac{12-2}{4} \\ &= \frac{5}{2}.\end{aligned}$$

To find δ we multiply both sides by $z-1$ and then we find the residue at 1:

$$\begin{aligned}\delta &= \operatorname{Res}_1 \frac{2z^3 - z^2 + 2z - 2}{(z^2+1)(z-1)} \\ &= \lim_{z\to 1} \frac{2z^3 - z^2 + 2z - 2}{z^2+1} \\ &= \frac{1}{2}.\end{aligned}$$

It follows that

$$\frac{z^6}{(z^2+1)(z-1)^2} = z^2+2z+2-\frac{1}{4(z-i)}-\frac{1}{4(z+i)}+\frac{5}{2(z-1)}+\frac{1}{2(z-1)^2},$$

6. Consider the biholomorphic map

$$\alpha\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \alpha(z) = Rz.$$

The composition $g = h \circ \alpha$ is a continuous funcion on the unit circle. It follows that

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2} g(e^{i\theta})\,\mathrm{d}\phi$$

is a harmonic function on the unit disk Δ with a continuous extension to the closed unit disk whose restriction ot the unit circle is g.
The inverse of α is the map

$$\beta\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \beta(z) = \frac{z}{R}.$$

As β is holomorphic the function

$$u(r,\theta) = v(r/R,\theta)$$

is then a harmonic function on the open disk U with a continuous extension to the boundary where it is equal to $h(Re^{i\theta})$.

We have

$$\begin{aligned}u(r,\theta) &= v(r/R,\theta)\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-(r/R)^2}{1-2(r/R)\cos(\phi-\theta)+(r/R)^2}g(e^{i\theta})\,\mathrm{d}\phi\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\phi-\theta)+r^2}h(Re^{i\theta})\,\mathrm{d}\phi\end{aligned}$$

Challenge Problems: (Just for fun)

7. (a) Show that every biholomorphic map of $U = \mathbb{C} - \{a_1, a_2, \ldots, a_n\}$, the complex plane punctured at finitely many points, is a Möbius transformation that permutes the points of

$$\{\, a_1, a_2, \ldots, a_n, \infty \,\}.$$

(b) Find the biholomorphic maps of $U = \mathbb{C} - \{0, 1\}$.
(c) Find the biholomorphic maps of $U = \mathbb{C} - \{-1, 0, 1\}$.
(d) Find the biholomorphic maps of $U = \mathbb{C} - \{-1, 0, 2\}$.
8. Let $f\colon \Delta \longrightarrow \Delta$ be a holomorphic map that is not biholomorphic. Show that if f has a fixed point a and f_n is the nth iterate of f (that is, compose f with itself n times) then the sequence of points

$$b \qquad f_1(b) = f(b) \qquad f_2(b) = f(f(b)) \qquad \ldots$$

converges to a, for any $b \in \Delta$.