MODEL ANSWERS TO THE SIXTH HOMEWORK

1. Note that f is not constant by assumption. In particular $|f(z)| < M$ be the maximum principle.
We first prove this for the unit disk. We have

$$f\colon \Delta \longrightarrow \Delta$$

and f has a zero of order m at 0. Schwarz's Lemma implies that

$$|f(z)| \leq |z|.$$

Consider

$$g(z) = \frac{f(z)}{z}.$$

Then

$$g\colon \Delta \longrightarrow \Delta$$

and g has a zero of order $m-1$ at 0. It follows by induction that

$$|f(z)| \leq |z|^m.$$

Further equality holds if and only if

$$f(z) = \lambda z^m,$$

for some scalar λ, with $|\lambda| = 1$.
Now we use the functions α and β in lecture 16.

$$\alpha\colon z \longrightarrow Rz + a \qquad \text{and} \qquad \beta\colon z \longrightarrow z/M.$$

Given

$$f\colon U \longrightarrow \mathbb{C} \qquad \text{such that} \qquad |f(z)| \leq M$$

let

$$g = \beta \circ f \circ \alpha\colon \Delta \longrightarrow \Delta.$$

As f has a zero of order m at a, g is a holomorphic map with a zero of order m at 0. By what we already proved

$$|g(w)| \leq |w|^m.$$

Apply the inverse of β to both sides it follows that

$$|(f \circ \alpha)(w)| \leq M^m |w|^m.$$

Pick $z \in U$. If we put

$$w = \frac{z-a}{R},$$

then $w \in \Delta$ and $\alpha(w) = z$. We have

$$\begin{aligned}|f(z)| &= |f(\alpha(w))| \\ &\leq M^m |w|^m \\ &= \frac{M^m}{R^m}|z-a|^m.\end{aligned}$$

Now suppose we have equality at some point not equal to a. Then we have equality for g at some point other than 0. But then

$$g(w) = \lambda w^m,$$

for some λ of modulus 1. In this case

$$\begin{aligned}f(z) &= g(w) \\ &= (M^m \lambda) w^m \\ &= \frac{M^m \lambda}{R^m}(z-a)^m.\end{aligned}$$

2. $\psi\colon \Delta \longrightarrow \Delta$ is a biholomorphic map taking a to 0. The composition

$$g = f \circ \psi\colon \Delta \longrightarrow \Delta$$

is a holomorphic map which has a zero of order m at zero. Thus

$$|g(w)| \leq |w|^m$$

by Question 1. If $z \in \Delta$ then we may find $w \in \Delta$ such that $\psi(w) = z$. In this case

$$\begin{aligned}|f(z)| &= |f(\psi(w))| \\ &= |g(w)| \\ &\leq |w|^m \\ &= |\psi(z)|^m.\end{aligned}$$

It follows that

$$\begin{aligned}|f(0)| &\leq |\psi(0)|^m \\ &= |a|^m.\end{aligned}$$

3. Suppose that $f(z)$ is nowhere zero. Then the function

$$p\colon \Delta \longrightarrow \mathbb{C}$$

given by

$$p(z) = \frac{1}{f(z)}$$

is holomorphic on the closed unit disk. Applying the maximum principle to the closed unit disk we see that $|p(z)|$ achieves its maximum at a point a on the circle $|z| = 1$. We have

$$\begin{aligned} 1 &= \frac{1}{|f(0)|} \\ &= \left|\frac{1}{f(0)}\right| \\ &= |p(0)| \\ &< |p(a)| \\ &= \left|\frac{1}{f(a)}\right| \\ &= \frac{1}{|f(a)|} \\ &< 1, \end{aligned}$$

which is not possible.
Thus $f(z)$ is zero somewhere in the unit disk.
Let

$$g\colon \Delta \longrightarrow \mathbb{C}$$

be given by

$$g(z) = \frac{f(z)}{M}.$$

Note that

$$|g(z)| < 1 \qquad \text{on} \qquad |z| = 1.$$

If a is a zero of f then it is also a zero of g and so by Question 2 we have

$$\begin{aligned} \frac{1}{M} &= \frac{|f(0)|}{M} \\ &= |g(0)| \\ &< |a|. \end{aligned}$$

4. One direction is clear. If $f(z)$ is a finite Blaschke product then $f(z)$ is holomorphic on the closed unit disk, so that it is certainly holomorphic on Δ and continuous on the closed unit disk and $|f(z)| = 1$ on $|z| = 1$, since it is a product of biholomorphic maps of the unit disk to itself.
Now suppose that $f(z)$ is holomorphic on Δ, continuous on the closed unit disk and $|f(z)| = 1$ on the circle $|z| = 1$. Note that $f(z)$ has only finitely many zeroes since if it had infinitely many zeroes they would accumulate on $|z| = 1$, contradicting the fact that $|f(z)| = 1$ on $|z| = 1$.
Let n be the number of zeroes.

Suppose first that $n = 0$, that $f(z)$ is nowhere zero on Δ. We want to show that $f(z)$ is constant. If not then $f(0) = b \in \Delta$. Consider

$$g \colon \Delta \longrightarrow \mathbb{C}$$

given by

$$g(z) = \frac{f(z)}{b}.$$

Then $g(0) = 1$ and if $|z| = 1$ then we have

$$\begin{aligned} |g(z)| &= \frac{|f(z)|}{|b|} \\ &= \frac{1}{|b|} \\ &> 1. \end{aligned}$$

Question 3 implies that $g(z)$ has a zero inside Δ. But a zero of g is a zero of f, which is not possible.

It follows that $f(z) = \lambda$ is a constant. As $|f(z)| = 1$ it follows that $|\lambda| = 1$ so that $\lambda = e^{i\varphi}$, where $\phi \in [0, 2\pi)$.

Now suppose that $n > 0$. Let $a_1, a_2, \dots, a_n$ be the zeroes of $f(z)$, repeated according to multiplicity. Let

$$B(z) = \left(\frac{z - a_1}{1 - \bar{a}_1 z}\right)\left(\frac{z - a_2}{1 - \bar{a}_2 z}\right)\dots\left(\frac{z - a_n}{1 - \bar{a}_n z}\right)$$

and consider

$$g \colon \Delta \longrightarrow \mathbb{C}$$

given by

$$g(z) = \frac{f(z)}{B(z)}.$$

A priori $g(z)$ is a meromorphic function. However, since every zero of $B(z)$ is matched by a zero of $f(z)$, it follows that $g(z)$ is holomorphic. Similarly $g(z)$ has no zeroes in the unit disk. Note that $g(z)$ extends to a continuous function on the closed unit disk and that on $|z| = 1$ we have

$$\begin{aligned} |g(z)| &= \left|\frac{f(z)}{B(z)}\right| \\ &= \frac{|f(z)|}{|B(z)|} \\ &= 1. \end{aligned}$$

As $g(z)$ is nowhere zero on the unit disk it follows that

$$g(z) = e^{i\varphi}$$

by what we already proved. But then $f(z)$ is a finite Blaschke product.
5. There are two ways to proceed. For the first observe that $f(z)$ is a rational function and so it is a meromorphic function. The denominator is zero at $\pm\sqrt{3}i$ and so f is a holomorphic function on Δ which extends to a continuous function on the circle $|z| = 1$.
If $z = e^{i\theta}$ is a point on the unit circle then

$$\begin{aligned} |1 + 3(e^{i\theta})^2| &= |1 + 3(e^{2i\theta})| \\ &= |1 + 3(e^{-2i\theta})| \\ &= |e^{2i\theta} + 3|. \end{aligned}$$

Thus $|f(z)| = 1$ on the unit circle. It follows by Question 4 that f is a finite Blaschke product.
For the second we just find an explicit representation of $f(z)$ as a finite Blaschke product. The zeroes of $f(z)$ are at

$$a_1 = \frac{i}{\sqrt{3}} \qquad \text{and} \qquad a_2 = -\frac{i}{\sqrt{3}}.$$

We have

$$\begin{aligned} \left(\frac{z - \frac{i}{\sqrt{3}}}{1 + \frac{i}{\sqrt{3}}z}\right)\left(\frac{z + \frac{i}{\sqrt{3}}}{1 - \frac{i}{\sqrt{3}}z}\right) &= \left(\frac{\sqrt{3}z - i}{\sqrt{3} + iz}\right)\left(\frac{\sqrt{3}z + i}{\sqrt{3} - iz}\right) \\ &= \frac{(\sqrt{3}z - i)(\sqrt{3}z + i)}{(\sqrt{3} + iz)(\sqrt{3} - iz)} \\ &= \frac{3z^2 + 1}{3 + z^2} \\ &= f(z). \end{aligned}$$

6. We first reduce to the unit disk and then we follow the proof of Schwarz's Lemma. Consider the function

$$g(z) = f(3z)\colon \Delta \longrightarrow \Delta.$$

Then

$$g(\pm 1/3) = 0 \qquad \text{and} \qquad g(\pm i/3) = 0.$$

We want to calculate the maximum value of $|g(0)|$. Consider the finite Blaschke product

$$B(z) = \left(\frac{z - \frac{1}{3}}{1 - \frac{z}{3}}\right)\left(\frac{z + \frac{1}{3}}{1 + \frac{z}{3}}\right)\left(\frac{z - \frac{i}{3}}{1 + \frac{iz}{3}}\right)\left(\frac{z + \frac{i}{3}}{1 - \frac{iz}{3}}\right).$$

Consider the function

$$h(z) = \frac{g(z)}{B(z)}.$$

This is a meromorphic function on the unit disk. As g is zero at the zeroes of B, which are all simple, it follows that h is a holomorphic function on the unit disk. Consider a circle of radius $r \in (0,1)$. If $|z| = r$ then

$$\begin{aligned} |h(z)| &= \left|\frac{g(z)}{B(z)}\right| \\ &= \frac{|g(z)|}{|B(z)|} \\ &\leq \frac{1}{r}. \end{aligned}$$

It follows by the maximum principle that

$$|h(z)| \leq \frac{1}{r}$$

on the open disk of radius r. Taking the limit as r approaches one we see that $|h(z)| \leq 1$ on the unit disk. Further equality holds if and only if $h(z) = \lambda$ is a constant of modulus 1.
In particular $|h(0)| \leq 1$ with equality if and only if $h(z) = e^{i\varphi}$. Thus

$$\begin{aligned} |f(0)| &= |g(0)| \\ &\leq |B(0)| \\ &= \frac{1}{3^4} \\ &= \frac{1}{81}, \end{aligned}$$

with equality if and only if

$$f(z) = e^{i\varphi} B(z/3).$$

7. We first consider the case $z_0 = r > 0$ and $z_1 = -r$. Given f let

$$g\colon \Delta \longrightarrow \mathbb{C}$$

be the holomorphic map

$$g(z) = \frac{f(z) - f(-z)}{2}.$$

We have

$$\begin{aligned} |g(r) - g(-r)| &= \left|\frac{f(r) - f(-r)}{2} - \frac{f(-r) - f(r)}{2}\right| \\ &= |f(r) - f(-r)|. \end{aligned}$$

Note that $g(0) = 0$ and

$$\begin{aligned} |g(z)| &= \left| \frac{f(z) - f(-z)}{2} \right| \\ &\leq \frac{1}{2} \left(|f(z)| + |f(-z)| \right) \\ &< 1. \end{aligned}$$

If we apply Schwarz's Lemma to $g(z)$ then we get $|g(z)| \leq |z|$. Thus

$$\begin{aligned} |g(r) - g(-r)| &\leq |g(r)| + |g(-r)| \\ &\leq r + r \\ &= 2r. \end{aligned}$$

If we have equality than

$$|f(z)| \geq |z| \qquad \text{for all} \qquad z \in \Delta.$$

Suppose that $f(z)$ is nowhere zero. Then

$$p(z) = \frac{1}{f(z)}$$

is holomorphic on Δ and

$$|p(z)| \leq \frac{1}{|z|}.$$

Applying the maximum principle on the circle of radius r we see that

$$|p(z)| \leq \frac{1}{r}.$$

Letting r go to one we get

$$|p(z) \leq 1.$$

But then

$$|f(z)| \geq 1,$$

which is not possible. Thus $f(z)$ has a zero somewhere. As

$$f(z) \geq |z|$$

we must have $f(0) = 0$. Schwarz's Lemma then implies that $f(z) = \lambda z$ for some scalar λ such that $|\lambda| = 1$.

Now suppose z_0 and z_1 are general. Let $\alpha\colon \Delta \longrightarrow \Delta$ be any biholomorphic map with inverse β and let $w_i = \alpha(z_i)$, $i = 0, 1$. If f maximises $|f(z_0) - f(z_1)|$ then $g = f \circ \beta$ maximises

$$|g(w_1) - g(w_0)| = |f(z_0) - f(z_1)|.$$

Consider the biholomorphic map α of Δ given by

$$z \longrightarrow \frac{z - z_0}{1 - \bar{z}_0 z}.$$

α sends z_0 to 0. If we apply a rotation to $\alpha(z_1)$ we may assume that $z_1 = x$ is a positive real.
If we use the biholomorphic map

$$z \longrightarrow \frac{z - r}{1 - rz}$$

to move 0 to $-r$ and x to r then we have

$$\frac{x - r}{1 - rx} = r \qquad \text{so that} \qquad xr^2 - 2r + x = 0.$$

Solving for r gives

$$\frac{2 \pm \sqrt{4 - 4x^2}}{2x} = \frac{1 \pm \sqrt{1 - x^2}}{x}.$$

We want the negative square root

$$r = \frac{1 - \sqrt{1 - x^2}}{x}.$$

Thus the maximum value is

$$\frac{2 - 2\sqrt{1 - x^2}}{x} \qquad \text{where} \qquad x = \left| \frac{z_1 - z_0}{1 - \bar{z}_0 z_1} \right|.$$

8. (a) There are many possibilities. One is

$$\alpha(z) = \frac{i(z + 1)}{1 - z}.$$

This sends 1 to ∞, -1 to 0 and i to -1. So three points of the unit circle go to three points of the real line. As a Möbius transformation take lines and circles to line and circles, it follows that this transformation takes the unit circle to the real axis. As 0 is sent to i it follows the unit disk is carried to the upper half plane.
(b) It is convenient to state an auxiliary result that we will use a little bit later. Consider the extended real line $\mathbb{R} \cup \{\infty\}$. Given any three distinct points α, β and γ of the extended real line, so that α, β and γ are either real numbers or ∞, there is a unique map

$$f \colon \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R} \cup \{\infty\}$$

of the extended real line to itself, of the form

$$f(x) = \frac{ax + b}{cx + d}$$

where a, b, c and d are real numbers and $ad - bc = \pm 1$.

We follow the same lines of proof as for the complex number. As the composition of Möbius transformations is a Möbius transformation we can prove this in stages. We want to send γ to infinity. We may assume $\gamma \neq \infty$. In this case we take $a = 0$, $b = c = 1$ and $d = -\gamma$. From now on we want to fix ∞, so we look at transformations of the form

$$x \longrightarrow ax + b$$

If we put $b = \alpha$ and $a = 1$ then we send 0 to α. Now we want to fix both 0 and ∞. This means we have a transformation of the form

$$x \longrightarrow ax$$

If we put $a = \beta$ then we send 1 to β. We already proved that there is at most one Möbius transformation with complex coefficients sending 0, 1 and ∞ to α, β and γ and so uniqueness is clear. If $ad - bc > 0$ and we multiply top and bottom by the square root of the reciprocal we are reduced to the case $ad - bc = 1$. If $ad - bc < 0$ by a similar trick we are reduced to $ad - bc = -1$.
Let $f \colon \mathbb{H} \longrightarrow \mathbb{H}$ be a biholomorphic map. Let

$$\beta(z) = \frac{z-1}{z+1}$$

be the inverse of the Möbius transformation α. Then

$$g = \beta \circ f \circ \alpha \colon \Delta \longrightarrow \Delta$$

is a holomorphic map from the unit disk to the unit disk. If f_0 is the inverse of f then $g_0 = \beta \circ f_0 \circ \alpha$ is the inverse of g. As g_0 is holomorphic g is biholomorphic. It follows that g is a Möbius transformation. From the equation $g = \beta \circ f \circ \alpha$ we get $f = \alpha \circ g \circ \beta$. But then f is a Möbius transformation.
Thus every biholomorphic map of the disk to itself is a Möbius transformation. g sends to the unit circle to the unit circle. As α sends the unit circle to the real axis, it follows that f sends the real axis to the real axis.
Consider the image of 0, 1 and ∞. We get three real numbers α, β and γ. There is a unique Möbius transformation which sends 0, 1 and ∞ to α, β and γ. As we already constructed one Möbius transformation with this property it must be the unique one and so

$$f(z) = \frac{az+b}{cz+d}$$

where a, b, c and d are real and $ab - bc = \pm 1$. Consider

$$\begin{aligned} f(i) &= \frac{ai+b}{ci+d} \\ &= \frac{(ai+b)(-ci+d)}{c^2+d^2} \\ &= \frac{ac+bd+i(ad-bc)}{c^2+d^2}. \end{aligned}$$

By assumption $f(i) \in \mathbb{H}$, so that the imaginary part $ad - bc > 0$. Thus $ad - bc = 1$.

(c) If $f\colon \mathbb{H} \longrightarrow \Delta$ is a biholomorphic map of the upper half plane to the unit disk then $f \circ \alpha\colon \Delta \longrightarrow \Delta$ is a biholomorphic map of the upper half plane to itself, where α is the Möbius transformation introduced in (a). As birational maps of the unit disk are Möbius transformations it follows that $f \circ \alpha$ is a Möbius transformation. Precomposing with the inverse of β and using the fact that the composition of Möbius transformations is a Möbius transformation, we see that f is a Möbius transformation.

As f is biholomorphic there is a point $a \in \mathbb{H}$ mapping to 0. Thus f must have the shape

$$f(z) = \frac{z-a}{cz+d}.$$

The point ∞ must map to a point $e^{i\varphi}$ of the unit circle. Thus

$$f(z) = e^{i\varphi}\frac{z-a}{z+d}.$$

The factor $e^{i\varphi}$ obviously corresponds to a rotation. Suppose that we could find two choices for d, d_1 and d_2, giving f_1 and f_2. The composition

$$f_1 \circ f_2^{-1}\colon \Delta \longrightarrow \Delta$$

is a biholomorphic map that fixes the origin. It is therefore a rotation and it is then easy to see that $d_1 = d_2$.

If $d = -\bar{a}$ then it is easy to see that any real number $z = x$ has the same distance to a as to $-\bar{a}$.

Hence every biholomorphic map of the upper half plane $\mathbb{H}$ to the unit disk Δ has the form

$$z \longrightarrow e^{i\varphi}\frac{z-a}{z-\bar{a}} \qquad \text{where} \qquad \operatorname{Im} a > 0, \varphi \in [0, 2\pi).$$

(d) a is the inverse image of f. The derivative of f is

$$f'(z) = e^{i\varphi}\frac{a-\bar{a}}{(z-\bar{a})^2}.$$

Thus

$$f'(0) = e^{i\varphi}\frac{a - \bar{a}}{\bar{a}^2}.$$

It follows that

$$e^{i\varphi} = f'(0)\frac{a - \bar{a}}{\bar{a}^2}.$$

It follows that we can recover φ as the argument of the RHS. As the RHS is determined by f, it follows that we can recover φ from f.