# MODEL ANSWERS TO THE FIFTH HOMEWORK

1. (a) Let $u(x,y) = x^2 - y^2$. We have

$$\frac{\partial u}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -2y.$$

It follows that

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= 2 - 2 \\ &= 0.\end{aligned}$$

Thus $u$ is harmonic.
We look for a harmonic conjugate. We have to solve the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = 2x \qquad \text{and} \qquad \frac{\partial v}{\partial x} = 2y.$$

We integrate the first equation with respect to $y$:

$$v(x,y) = 2xy + h(x).$$

Here the constant of integration $h(x)$ is an arbitrary function of $x$. We put the known value for $v$ into the second equation:

$$h'(x) = 0.$$

Thus $v(x,y) = 2xy$ is a harmonic conjugate of $u$. The corresponding holomorphic function is

$$f(z) = z^2.$$

(b) Let $u(x,y) = xy + 3x^2y - y^3$. We have

$$\frac{\partial u}{\partial x} = y + 6xy \qquad \text{and} \qquad \frac{\partial u}{\partial y} = x + 3x^2 - 3y^2.$$

It follows that

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= 6y - 6y \\ &= 0.\end{aligned}$$

Thus $u$ is harmonic.

We look for a harmonic conjugate. We have to solve the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = y + 6xy \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -x - 3x^2 + 3y^2.$$

We integrate the first equation with respect to $y$:

$$v(x, y) = \frac{y^2}{2} + 3xy^2 + h(x).$$

We put the known value for $v$ into the second equation:

$$h'(x) = -x - 3x^2.$$

Integrating with respect to $x$ we get

$$h(x) = -\frac{x^2}{2} - x^3.$$

Thus

$$v(x, y) = \frac{y^2}{2} + 3xy^2 - \frac{x^2}{2} - x^3.$$

is a harmonic conjugate of $u$. We already saw that

$$-i\frac{z^2}{2}$$

is a holomorphic function whose real part is $xy$. The function

$$z^3$$

has real and imaginary parts

$$x^3 - 3xy^2 \qquad \text{and} \qquad 3x^2y - y^3.$$

Thus

$$f(z) = -i\frac{z^2}{2} + iz^3$$

is a holomorphic function with real part $u$ and imaginary part $v$.
(c) Let $u(x, y) = \sinh x \sin y$. We have

$$\frac{\partial u}{\partial x} = \cosh x \sin y \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \sinh x \cos y.$$

It follows that

$$\begin{aligned}\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ &= \sinh x \sin y - \sinh x \sin y \\ &= 0.\end{aligned}$$

Thus $u$ is harmonic.

We look for a harmonic conjugate. We have to solve the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = \cosh x \sin y \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\sinh x \cos y.$$

We integrate the first equation with respect to $y$:

$$v(x, y) = -\cosh x \cos y + h(x).$$

We put the known value for $v$ into the second equation:

$$h'(x) = 0.$$

Thus

$$v(x, y) = -\cosh x \cos y.$$

is a harmonic conjugate of $u$.
Note that

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

see Homework 2 of Math 120A. Thus

$$\cos iz = \cos y \cosh x + i \sin y \sinh x$$

Hence the function

$$f(z) = -i \cos iz$$

is a holomorphic function with real part $u$ and imaginary part $v$.
(d) Let $u(x, y) = \frac{x}{x^2+y^2}$. We have

$$\frac{\partial u}{\partial x} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}$$

It follows that

$$\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&= \frac{-2x(x^2 + y^2)^2 + 4x(x^2 - y^2)(x^2 + y^2)}{(x^2 + y^2)^4} - \frac{2x(x^2 + y^2)^2 - *xy^2(x^2 + y^2)}{(x^2 + y^2)^4} \\
&= \frac{4x}{(x^2 + y^2)^3} \left( -(x^2 + y^2) + (x^2 - y^2) + 2y^2 \right) \\
&= 0.
\end{aligned}$$

Thus $u$ is harmonic.
We look for a harmonic conjugate. We have to solve the Cauchy-Riemann equations:

$$\frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}$$

We integrate the second equation with respect to $x$:

$$v(x,y) = -\frac{y}{x^2+y^2} + h(y).$$

We put the known value for $v$ into the first equation:

$$h'(y) = 0.$$

Thus

$$v(x,y) = -\frac{y}{x^2+y^2}$$

is a harmonic conjugate of $u$. In fact

$$f(z) = \frac{1}{z}$$

is a holomorphic function with real part $u$ and imaginary part $v$.
2. By assumption $f = u+iv$ is holomorphic. In this case $-if = v - iu$ is holomorphic. As $v$ is the real part of $if$ and $-u$ is the imaginary part, it follows that $-u$ is a harmonic conjugate of $V$.
3. (a) One way to do this is to write down the given expression and use the chain rule to manipulate it to the Laplacian in Cartesian coordinates. This approach is a little bit unsatisfying as it is then not obvious how one would know the form of the Laplacian in polar coordinates in the first place. We give another approach. Either method involves quite a bit of computation.
We want to express $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ in terms of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. It is easiest to do the opposite and then invert a matrix.
We have

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

Thus

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial u}{\partial x}\sin\theta + r\frac{\partial u}{\partial y}\cos\theta.$$

In matrix form we get

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.$$

As the determinant of the $2\times 2$ matrix is $r$ we can invert the matrix to get

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}$$

It follows that

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta \qquad \text{and} \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta.$$

We have

$$
\begin{aligned}
u_{xx} &= \frac{\partial}{\partial x}\left(u_r \cos\theta - \frac{1}{r}u_\theta \sin\theta\right) \\
&= \cos\theta \frac{\partial}{\partial r}\left(u_r \cos\theta - \frac{1}{r}u_\theta \sin\theta\right) - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\left(u_r \cos\theta - \frac{1}{r}u_\theta \sin\theta\right) \\
&= \cos^2\theta u_{rr} + \frac{1}{r^2}\cos\theta\sin\theta u_\theta \frac{1}{r^2}\cos\theta\sin\theta u_\theta - \frac{1}{r}\cos\theta\sin\theta u_{r\theta} - \frac{1}{r}\cos\theta\sin\theta u_{\theta r} - \frac{1}{r}\sin^2\theta u_r + \\
&= \cos^2\theta u_{rr} + \frac{2}{r^2}\cos\theta\sin\theta u_\theta - \frac{2}{r}\cos\theta\sin\theta u_{r\theta} + \frac{1}{r}\sin^2\theta u_r + \frac{1}{r^2}\sin^2\theta u_{\theta\theta}
\end{aligned}
$$

and

$$
\begin{aligned}
u_{yy} &= \frac{\partial}{\partial y}\left(u_r \sin\theta + \frac{1}{r}u_\theta \cos\theta\right) \\
&= \sin\theta \frac{\partial}{\partial r}\left(u_r \sin\theta + \frac{1}{r}u_\theta \cos\theta\right) + \frac{1}{r}\cos\theta \frac{\partial}{\partial \theta}\left(u_r \sin\theta + \frac{1}{r}u_\theta \cos\theta\right) \\
&= \sin^2\theta u_{rr} - \frac{1}{r^2}\cos\theta\sin\theta u_\theta + \frac{1}{r}\cos\theta\sin\theta u_{r\theta} + \frac{1}{r}\cos\theta\sin\theta u_{\theta r} + \frac{1}{r}\cos^2\theta u_r + \frac{1}{r^2}\cos^2\theta u_{\theta\theta} - \\
&= \sin^2\theta u_{rr} - \frac{2}{r^2}\cos\theta\sin\theta u_\theta + \frac{2}{r}\cos\theta\sin\theta u_{r\theta} + \frac{1}{r}\cos^2\theta u_r + \frac{1}{r^2}\cos^2\theta u_{\theta\theta}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Delta u &= u_{xx} + u_{yy} \\
&= (\cos^2\theta + \sin^2\theta)u_{rr} + (\cos^2\theta + \sin^2\theta)\frac{1}{r}u_r + (\cos^2\theta + \sin^2\theta)\frac{1}{r^2}u_{\theta\theta} \\
&= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.
\end{aligned}
$$

Hence Laplace's equation in polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

(b) Consider

$$u(r,\theta) = \ln r.$$

Then

$$\frac{\partial u}{\partial r} = \frac{1}{r},$$

so that

$$
\begin{aligned}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2} + \frac{\partial^2 u}{\partial \theta^2} &= -\frac{1}{r^2} + \frac{1}{r}\frac{1}{r} \\
&= 0.
\end{aligned}
$$

It follows that $\ln|z|$ is harmonic on the punctured plane $\mathbb{C} \setminus \{0\}$.

(c) $v = (x, y) = \mathrm{Arg}(z)$ is a harmonic conjugate of $\ln|z|$ on the complex plane $V = \mathbb{C} \setminus (-\infty, 0]$ minus the non-positive reals. The function

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$$

is the principal branch of the logarithm.

Suppose that $w$ is a harmonic conjugate for $u$ on the whole punctured plane. Then $v - w$ is a constant function on $V$. But then if we can extend $w$ then we can extend $v$, which is not possible.

(d) We have

$$\frac{\partial u}{\partial r} = \frac{\theta}{r} \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = \ln r.$$

It follows that

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2} + \frac{\partial^2 u}{\partial \theta^2} &= -\frac{\theta}{r^2} + \frac{\theta}{r^2} + \frac{1}{r^2} \cdot 0 \\ &= 0.\end{aligned}$$

Thus $\theta \ln r$ is harmonic. Recall the polar form of the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

In our case these reduce to

$$\frac{\partial v}{\partial \theta} = \theta \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{\ln r}{r}.$$

If we integrate the first equation with respect to $\theta$ we get

$$v(r, \theta) = \frac{\theta^2}{2} + h(r).$$

Plugging this into the second equation gives

$$h'(r) = -\frac{\ln r}{r}.$$

One solution to this differential equation is:

$$h(r) = \frac{-(\ln r)^2}{2}.$$

Thus

$$v(r, \theta) = \frac{\theta^2}{2} - \frac{(\ln r)^2}{2}$$

is a harmonic conjugate of $u(r, \theta) = \theta \ln r$.

Let

$$f(z) = -\frac{i(\mathrm{Log}\, z)^2}{2}.$$

Then $f(z)$ is a holomorphic function with real part $u$ and imaginary part $v$.

4. Pick a reference point in the annulus. A natural choice is $r_0 \in (a, b)$. We solve the Cauchy-Riemann equations in polar form:

$$\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r} \qquad \text{and} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

on the annulus cut along $(a, b)$. We follow the procedure of lecture 13. Note that if you unwrap the annulus separating the two pieces of the cut then we get a rectangle.
We start with the second equation and we integrate with respect to $r$:

$$v(r, \theta) = -\int_{r_0}^{r} \frac{1}{t}\frac{\partial u}{\partial \theta}(t, \theta)\,\mathrm{d}t + h(\theta).$$

If we plug this value into the first equation then we get

$$\begin{aligned}
\frac{\partial u}{\partial r}(r, \theta) &= -\frac{1}{r}\frac{\partial}{\partial \theta}\int_{r_0}^{r} \frac{1}{t}\frac{\partial u}{\partial \theta}(t, \theta)\,\mathrm{d}t + \frac{1}{r}h'(\theta) \\
&= -\frac{1}{r}\int_{r_0}^{r} \frac{1}{t}\frac{\partial^2 u}{\partial \theta^2}(t, \theta)\,\mathrm{d}t + \frac{1}{r}h'(\theta) \\
&= \frac{1}{r}\int_{r_0}^{r} t\frac{\partial^2 u}{\partial r^2}(t, \theta) + \frac{\partial u}{\partial r}(t, \theta)\,\mathrm{d}t + \frac{1}{r}h'(\theta) \\
&= \frac{1}{r}\int_{r_0}^{r} \frac{\partial}{\partial r}\left(t\frac{\partial u}{\partial r}\right)\,\mathrm{d}t + \frac{1}{r}h'(\theta) \\
&= \frac{\partial u}{\partial r}(r, \theta) - \frac{r_0}{r}\frac{\partial u}{\partial r}(r_0, \theta) + \frac{1}{r}h'(\theta).
\end{aligned}$$

Hence

$$h'(\theta) = r_0\frac{\partial u}{\partial r}(r_0, \theta).$$

Integrating both sides with respect to $\theta$ we get

$$h(\theta) = r_0\int_{0}^{\theta} \frac{\partial u}{\partial r}(r_0, \phi)\,\mathrm{d}\phi,$$

up to a constant.
Thus

$$v(r, \theta) = -\int_{r_0}^{r} \frac{1}{t}\frac{\partial u}{\partial \theta}(t, \theta)\,\mathrm{d}t + r_0\int_{0}^{\theta} \frac{\partial u}{\partial r}(r_0, \phi)\,\mathrm{d}\phi,$$

up to a constant.
What is the obstruction to extending this function to the whole annulus? The problem is to extend $v$ across the cut and the only reason we might not be able to do this is because the value of $v$ as we approach from below the cut is not equal to its value as we approach from above.

Note the problem is to control what happens as $\theta$ approaches $2\pi$. This is controled by the second term in the integral

$$r_0 \int_0^{2\pi} \frac{\partial u}{\partial r}(r_0, \phi)\, \mathrm{d}\phi.$$

If we want a continuous function across the cut then we want the value of this integral to be zero.
Note that if

$$w(r, \theta) = \ln r$$

then

$$\begin{aligned} r_0 \int_0^{2\pi} \frac{\partial w}{\partial r}(r_0, \phi)\, \mathrm{d}\phi &= r_0 \int_0^{2\pi} \frac{1}{r_0}\, \mathrm{d}\phi \\ &= \int_0^{2\pi} 1\, \mathrm{d}\phi \\ &= 2\pi. \end{aligned}$$

On the other hand, we already saw in Question 3 that $w$ is harmonic, so that it has a harmonic conjugate on the cut annulus. It follows that if we consider

$$u(r, \theta) - C \ln r$$

then it is harmonic and has a harmonic conjugate on the whole annulus, where

$$C = \frac{r_0}{2\pi} \int_0^{2\pi} \frac{\partial u}{\partial r}(r_0 e^{i\theta})\, \mathrm{d}\theta.$$

5. As $u \in [a, b]$ on the boundary $\partial U$, it follows that $u(x, y) \leq b$ on the boundary. As $u$ is continuous on the $U \cup \partial U$ it has a maximum somewhere. If this maximum is on $U$ then $u$ is constant or this maximum is on the boundary. Either way, $u(x, y) \leq b$.
Consider $-u$. This is a harmonic function such that $-u(x, y) \leq -a$ on the boundary. It follows that $-u(x, y) \leq -a$ on $U$. Thus $u(x, y) \geq a$ on $U$ (this conclusion is sometimes called the minimum principle). Thus $u(x, y) \in [a, b]$.
6. By the maximum principle, we just have to find the maximum over the boundary of the closed unit disk. By the triangle equality $z^n + \lambda$ is a maximum when $z^n$ and $\lambda$ are parallel. In this case the maximum is $r^n + \rho$.
$z^n$ is parallel to $\lambda$ if and only if

$$z^n = r^n e^{i\theta}$$

Therefore the maximum is achieved at

$$r\omega^k e^{\theta i/n} \qquad \text{where} \qquad 0 \leq k \leq n-1,$$

and

$$\omega = e^{2\pi i/n}$$

is a root of unity.

7. (a) Let

$$g(z) = \frac{1}{f(z)}.$$

As $f(z)$ is holomorphic and nowhere zero on $U$ it follows that $g(z)$ is holomorphic on $U$.

Suppose that $|f(z)| \geq m$ and $f(a) = m$. Let

$$M = \frac{1}{m}.$$

Then

$$\begin{aligned} |g(z)| &= \frac{1}{|f(z)|} \\ &\leq M, \end{aligned}$$

and $|g(a)| = M$. The strong maximum principle implies that $g(z)$ is constant. But then $f(z)$ is constant.

(b) There are two cases. If $f(z)$ is zero somewhere on the boundary then $|f(z)|$ is zero there and this is a minimum. Otherwise $g(z)$ extends to a continuous function on the boundary. In this case the maximum principle applied to $g$ implies that $g$ achieves its maximum somewhere on the boundary. This maximum is a minimum of $f(z)$ on the boundary.

8. Consider

$$g(z) = (z+1)^{-\epsilon} f(z),$$

where $\epsilon > 0$ is a real number.

We first make sense of the power of $(z+1)$. Start with the standard branch of the logarithm, where we cut out the non-positive reals, $(-\infty, 0]$ and we let

$$\log z = \ln|z| + i \arg(z) \qquad \text{where} \qquad \arg(z) \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

Using this branch of the logarithm we may define

$$z^{-\epsilon} = e^{-\epsilon \log z}$$

as a holomorphic function on

$$V = \mathbb{C} \setminus (-\infty, 0].$$

In particular this defines $g(z)$ as a holomorphic function on the half plane, $\operatorname{Re}(z) > -1$. Let $U$ be the open disk of radius $R$ centred at

the origin intersected with the right half plane. Then $g(z)$ extends continuously to the boundary of $U$.
The maximum principle implies that $|g(z)|$ achieves its maximum somewhere on the boundary of $U$. On the imaginary axis we have $|1+z| \geq 1$ and so

$$\begin{aligned} |g(z)| &= |(z+1)^{-\epsilon} f(z)| \\ &\leq |1|^{-\epsilon} |f(z)| \\ &\leq M. \end{aligned}$$

Suppose that $|f(z)| \leq M_0$. On the semicircle of radius $R$ we have

$$\begin{aligned} |g(z)| &= |(z+1)^{-\epsilon} f(z)| \\ &\leq \frac{M_0}{(R-1)^{\epsilon}}. \end{aligned}$$

If we fix $\epsilon$ and let $R$ go to infinity the last expression goes to zero. In particular we may assume that $|g(z)| \leq M$ on the boundary of $U$.
The maximum principle implies that $|g(z)| \leq M$ on $U$.
Suppose that there is a point $a$ in the right half plane where $|f(a)| = \mu > M$. Suppose that $r = |a|$. Note that

$$\lim_{\epsilon \to 0} |r+1|^{\epsilon} = 1.$$

Therefore we may pick $\epsilon > 0$ sufficiently small so that

$$\begin{aligned} |g(a)| &= |a+1|^{-\epsilon} |f(a)| \\ &= \frac{|f(a)|}{|a+1|^{\epsilon}} \\ &\geq \frac{\mu}{|r+1|^{\epsilon}} \\ &> M. \end{aligned}$$

As we already decided this is not possible, there is no such $a$ and it must in fact be the case that $|f(z)| \leq M$ for every $z$ in the right half plane.

**Challenge Problems:** (Just for fun)

9. Suppose that $p(z)$ is nowhere zero. Let $U$ be the open disk of radius $R$ centred at the origin. Question 7 (b) implies that if $m$ is the minimum of $|p(z)|$ on the circle of radius $R$ then

$$|p(z)| \geq m$$

on $U$.

If $R$ is sufficiently large then $m$ goes to infinity, which is clearly impossible. It follows that $p(z)$ is zero somewhere.