MODEL ANSWERS TO THE THIRD HOMEWORK

1. Let

$$f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}.$$

This has a pole at 0 and so we integrate around the indented contour

$$\gamma = \gamma_- + \gamma_0 + \gamma_+ + \gamma_2,$$

where γ_- goes from $-R$ to $-\rho$, γ_0 goes along the semicircle of radius ρ from $-\rho$ to ρ in the upper half plane, γ_+ goes from ρ to R and γ_2 goes back to $-R$ along the semicircle of radius R in the upper half plane.
As $f(z)$ is holomorphic on

$$U = \{\, z \in \mathbb{C} \mid \rho < |z| < R \,\} \cap \mathbb{H},$$

whose boundary is γ, Cauchy's theorem implies that

$$\int_\gamma \frac{e^{iaz} - e^{ibz}}{z^2}\, \mathrm{d}z = 0.$$

We estimate the integral of $f(z)$ on γ_2. For the maximum value M we have

$$\begin{aligned} \left| \frac{e^{iaz} - e^{ibz}}{z^2} \right| &= \frac{|e^{iaz} - e^{ibz}|}{|z^2|} \\ &\leq \frac{2}{R^2}. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \int_{\gamma_2} \frac{e^{iaz} - e^{ibz}}{z^2}\, \mathrm{d}z \right| &\leq LM \\ &\leq \frac{\pi R}{R^2} \\ &\leq \frac{\pi}{R}, \end{aligned}$$

which goes to zero as R goes to infinity.
Note that $f(z)$ has a simple pole at 0, since

$$e^{iaz} - e^{ibz} = i(a-b)z + \dots$$

has a simple zero. We can also use this to compute the residue:

$$\operatorname{Res}_0 f(z) = i(a-b).$$

It follows that

$$\lim_{\rho\to 0}\int_{\gamma_0}\frac{e^{iaz}-e^{ibz}}{z^2}\,\mathrm{d}z = -\pi i i(a-b)$$
$$= \pi(a-b).$$

If we let R to ∞ and ρ go to zero then the integral over γ_- and γ_+ approaches the Cauchy principal value of

$$\int_{-\infty}^{\infty}\frac{\cos(ax)-\cos(bx)}{x^2}\,\mathrm{d}x.$$

It follows that the Cauchy principal value of the integral above is $\pi(b-a)$. Taking real parts this implies that the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty}\frac{\cos(ax)-\cos(bx)}{x^2}\,\mathrm{d}x$$

is also $\pi(a-b)$. As the integrand

$$\frac{\cos(ax)-\cos(bx)}{x^2}$$

is even it follows

$$\int_{0}^{\infty}\frac{\cos(ax)-\cos(bx)}{x^2}\,\mathrm{d}x = \frac{\pi}{2}(b-a).$$

If we put $a=0$ and $b=2$ then we get

$$\begin{aligned}\int_{0}^{\infty}\frac{\sin^2 x}{x^2}\,\mathrm{d}x &= \frac{1}{2}\int_{0}^{\infty}\frac{1-\cos(2x)}{x^2}\,\mathrm{d}x\\ &= \frac{1}{2}\frac{\pi}{2}(2-0)\\ &= \frac{\pi}{2}.\end{aligned}$$

2. Let

$$f(z) = \frac{1}{\sqrt{z}(z^2+1)}.$$

We have to choose a branch of the logarithm to make sense of $f(z)$.
(i) We use the same branch of the logarithm as in lecture 7. We cut the complex plane along the negative imaginary axis:

$$V = \mathbb{C}\setminus\{\, iy \mid y\leq 0\,\}.$$

We then choose a branch of the logarithm

$$\log z = \ln|z| + i\arg z \qquad \text{where} \qquad \arg z\in(-\pi/2, 3\pi/2).$$

We use this to define

$$\sqrt{z} = e^{\log z/2}.$$

This makes $\sqrt{z}$ a holomorphic function on V.
We integrate along the same contour as in question 1. $f(z)$ has one isolated singularity as i. This is a simple pole and the residue is:

$$\begin{aligned}\operatorname{Res}_i f(z) &= \lim_{z\to i} \frac{z-i}{\sqrt{z}(z^2+1)} \\ &= \lim_{z\to i} \frac{1}{\sqrt{z}(z+i)} \\ &= \frac{1}{2i\sqrt{i}} \\ &= \frac{1}{2ie^{\pi i/4}} \\ &= \frac{1}{2i} e^{-\pi i/4}.\end{aligned}$$

The residue theorem gives

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}z}{\sqrt{z}(z^2+1)} &= 2\pi i \operatorname{Res}_i f(z) \\ &= \pi e^{-\pi i/4}.\end{aligned}$$

We estimate the integral of $f(z)$ on γ_2. For the maximum value M we have

$$\begin{aligned}\left|\frac{1}{\sqrt{z}(z^2+1)}\right| &= \frac{1}{\sqrt{z}(z^2+1)|} \\ &\leq \frac{1}{R^{1/2}(R^2-1)}.\end{aligned}$$

It follows that

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\mathrm{d}z}{\sqrt{z}(z^2+1)}\right| &\leq LM \\ &\leq \frac{\pi R}{R^{1/2}(R^2-1)} \\ &= \frac{\pi R^{1/2}}{R^2-1},\end{aligned}$$

which goes to zero as R goes to infinity.

Now we compute what happens over γ_0 as ρ goes to zero. We estimate the maximum value M of $|f(z)|$ over γ_0:

$$\begin{aligned} \left| \frac{1}{\sqrt{z}(z^2+1)} \right| &= \frac{1}{\sqrt{z}(z^2+1)|} \\ &\leq \frac{1}{\rho^{1/2}(1-\rho^2)}. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \int_{\gamma_2} \frac{\mathrm{d}z}{\sqrt{z}(z^2+1)} \right| &\leq LM \\ &\leq \frac{\pi\rho}{\rho^{1/2}(1-\rho^2)} \\ &= \frac{\pi\rho^{1/2}}{1-\rho^2}, \end{aligned}$$

which goes to zero as ρ goes to zero.

The integral over γ_+ is equal to

$$\int_{\gamma_+} \frac{\mathrm{d}z}{\sqrt{z}(z^2+1)} = \int_\rho^R \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)}$$

which goes to the value of the improper integral I we are trying to compute, as ρ goes to zero and R to infinity.

Finally, for the integral over γ_- we use the parametrisation

$$z = -x \qquad \text{where} \qquad x \in [\rho, R].$$

This traverses γ_- in the wrong direction.

$$\int_{\gamma_-} \frac{\mathrm{d}z}{\sqrt{z}(z^2+1)} = -i \int_\rho^R \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)}.$$

Note that the minus sign represents three minus signs; one as $\mathrm{d}z = -\mathrm{d}x$, one for the fact that we traverse γ_- in the wrong direction and one to move i from the denominator to the numerator.

If we Let R go to infinity and ρ go to zero then we get

$$(1-i)I = \pi e^{-\pi i/4}.$$

But then

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

(ii) We use the same branch of the logarithm as in lecture 8. We cut out the non-negative real axis.

$$V = \mathbb{C} \setminus \{\, x \mid x \geq 0 \,\}.$$

We are going to make a choice of $\log z$ with a cut along the positive real axis:

$$\log z = \ln|z| + i\arg z \qquad \text{where} \qquad \arg z \in (0, 2\pi).$$

We also use the same contour as the one used in lecture 8.
$f(z)$ has isolated singularities at $\pm i$. They are both simple poles. We already computed the residue at i,

$$\operatorname{Res}_i f(z) = \frac{1}{2i} e^{-\pi i/4}.$$

For the residue at $-i$ we have

$$\begin{aligned}
\operatorname{Res}_{-i} f(z) &= \lim_{z\to -i} \frac{z+i}{\sqrt{z}(z^2+1)} \\
&= \lim_{z\to -i} \frac{1}{\sqrt{z}(z-i)} \\
&= \frac{1}{-2i\sqrt{-i}} \\
&= -\frac{1}{2ie^{-\pi i/4}} \\
&= -\frac{1}{2i} e^{\pi i/4}.
\end{aligned}$$

The residue theorem gives

$$\begin{aligned}
\int_\gamma \frac{\mathrm{d}z}{\sqrt{z}(z^2+1)} &= 2\pi i \left(\operatorname{Res}_i f(z) + \operatorname{Res}_{-i} f(z)\right) \\
&= \pi \left(e^{-\pi i/4} + e^{\pi i/4}\right) \\
&= \pi\sqrt{2}.
\end{aligned}$$

The integral over γ_2 still goes to zero, since the upper bound we established in (i) is still valid and the length L doubled. Similarly the integral over γ_ρ goes to zero. The integral over γ_+ is the same as in (i):

$$\int_{\gamma_+} \frac{\mathrm{d}z}{\sqrt{z}(z^2+1)} = \int_\rho^R \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)}$$

which goes to the value of the improper integral I we are trying to compute, as ρ goes to zero and R to infinity.
Finally, for the integral over γ_- we use the parametrisation

$$z = x \qquad \text{where} \qquad x \in [\rho, R].$$

This traverses γ_- in the wrong direction.

$$\int_{\gamma_-} \frac{\mathrm{d}z}{\sqrt{z}(z^2+1)} = \int_\rho^R \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)}.$$

Note that the plus sign represents two minus signs; one going in the wrong direction and one for the fact that $\sqrt{z} = -\sqrt{x}$ just below the cut.
If we let R go to infinity and ρ go to zero then we get

$$2I = \pi\sqrt{2}.$$

But then

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

3. Let

$$f(z) = \frac{\log z}{(z^2+1)(z+1)},$$

where $\log z$ is the same branch of the logarithm as in 2 (ii). We integrate this around the keyhole contour of 2 (ii).
$f(z)$ has isolated singularities at $\pm i$ and -1, which are all simple. We compute the residues. We have

$$\begin{aligned}
\operatorname{Res}_i f(z) &= \lim_{z\to i} \frac{(z-i)\log z}{(z^2+1)(z+1)} \\
&= \lim_{z\to i} \frac{\log z}{(z+i)(z+1)} \\
&= \frac{\pi i/2}{(2i)(i+1)} \\
&= \frac{\pi(1-i)}{8},
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Res}_{-i} f(z) &= \lim_{z\to -i} \frac{(z+i)\log z}{(z^2+1)(z+1)} \\
&= \lim_{z\to -i} \frac{\log z}{(z-i)(z+1)} \\
&= \frac{3\pi i/2}{(-2i)(-i+1)} \\
&= -\frac{3\pi(1+i)}{8},
\end{aligned}$$

and finally

$$\begin{aligned}\operatorname{Res}_{-1} f(z) &= \lim_{z\to -1} \frac{(z+1)\log z}{(z^2+1)(z+1)} \\ &= \lim_{z\to -1} \frac{\log z}{(z^2+1)} \\ &= \frac{\pi i}{2}.\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}\int_\gamma \frac{\log z}{(z^2+1)(z+1)}\,\mathrm{d}z &= 2\pi i\left(\operatorname{Res}_i f(z) + \operatorname{Res}_{-i} f(z) + \operatorname{Res}_{-1} f(z)\right) \\ &= 2\pi i \frac{\pi}{8}\left((1-i) - 3(1+i) + 4i\right) \\ &= -2\pi i \frac{\pi}{4}.\end{aligned}$$

Next we show the integrals over γ_2 and γ_0 go to zero. As usual we have to estimate the largest value of $|f(z)|$. Over γ_2 we have

$$\begin{aligned}|f(z)| &= \frac{|\log z|}{|(z^2+1)(z+1)|} \\ &\leq \frac{\ln R + 2\pi}{(R^2-1)(R-1)}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\log z}{(z^2+1)(z+1)}\,\mathrm{d}z\right| &\leq LM \\ &\leq \frac{2\pi R(\ln R + 2\pi)}{(R^2-1)(R-1)},\end{aligned}$$

which goes to zero as R goes to infinity. Over γ_0 we have

$$\begin{aligned}|f(z)| &= \frac{|\log z|}{|(z^2+1)(z+1)|} \\ &\leq \frac{2\pi - \ln\rho}{(1-\rho^2)(1-\rho)}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\log z}{(z^2+1)(z+1)}\,\mathrm{d}z\right| &\leq LM \\ &\leq \frac{2\pi\rho(2\pi - \ln\rho)}{(1-\rho^2)(1-\rho)},\end{aligned}$$

which goes to zero as ρ goes to zero, since $\rho\ln\rho$ goes to zero.

The integral over γ_+ is equal to

$$\int_{\gamma_+} \frac{\log z}{(z^2+1)(z+1)}\,\mathrm{d}z = \int_\rho^R \frac{\ln x}{(x^2+1)(x+1)}\,\mathrm{d}x.$$

Finally, for the integral over γ_- we use the same parametrisation

$$z = x \qquad \text{where} \qquad x \in [\rho, R]$$

but with a different branch of the logarithm

$$\log z = \ln x + 2\pi i.$$

This traverses γ_- in the wrong direction, so we flip the sign.

$$\int_{\gamma_-} \frac{\log z}{(z^2+1)(z+1)}\,\mathrm{d}z = -\int_\rho^R \frac{\ln x}{(x^2+1)(x+1)}\,\mathrm{d}x - 2\pi i \int_\rho^R \frac{1}{(x^2+1)(x+1)}\,\mathrm{d}x.$$

Letting ρ go to zero and R go to infinity we get:

$$-2\pi i I = -2\pi i \frac{\pi}{4}.$$

Solving for I gives

$$\int_0^\infty \frac{1}{(x^2+1)(x+1)}\,\mathrm{d}x = \frac{\pi}{4}.$$

4. Let

$$f(z) = \frac{\sqrt[3]{z}}{(z+a)(z+b)}$$

We use the same branch of the logarithm and keyhole contour as in 2 (ii). $f(z)$ has isolated singularities at $-a$ and $-b$. They are both simple poles. We calculate the residues there:

$$\begin{aligned}\operatorname{Res}_{-a} f(z) &= \lim_{z\to -a} \frac{\sqrt[3]{z}}{z+b}\\ &= \frac{\sqrt[3]{-a}}{-a+b}\\ &= \frac{e^{\pi i/3}\sqrt[3]{a}}{-a+b}.\end{aligned}$$

By symmetry we also get

$$\operatorname{Res}_{-b} f(z) = \frac{e^{\pi i/3}\sqrt[3]{b}}{a-b}.$$

The residue theorem implies that

$$\begin{aligned}\int_\gamma \frac{\sqrt[3]{z}}{(z+a)(z+b)}\,\mathrm{d}z &= 2\pi i\left(\mathrm{Res}_{-a} f(z) + \mathrm{Res}_{-b} f(z)\right)\\ &= 2\pi i\left(\frac{e^{\pi i/3}\sqrt[3]{a}}{-a+b} + \frac{e^{\pi i/3}\sqrt[3]{b}}{a-b}\right)\\ &= -2\pi i e^{\pi i/3}\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}.\end{aligned}$$

Next we show the integrals over γ_2 and γ_0 go to zero. As usual we have to estimate the largest value of $|f(z)|$. Over γ_2 we have

$$\begin{aligned}|f(z)| &= \frac{|\sqrt[3]{z}|}{|(z+a)(z+b)|}\\ &\leq \frac{R^{1/3}}{(R-a)(R-b)}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\sqrt[3]{z}}{(z+a)(z+b)}\,\mathrm{d}z\right| &\leq LM\\ &\leq \frac{2\pi R^{4/3}}{(R-a)(R-b)},\end{aligned}$$

which goes to zero as R goes to infinity. Over γ_0 we have

$$\begin{aligned}|f(z)| &= \frac{|\sqrt[3]{z}|}{|(z+a)(z+b)|}\\ &\leq \frac{\rho^{1/3}}{(a-\rho)(b-\rho)}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\sqrt[3]{z}}{(z+a)(z+b)}\,\mathrm{d}z\right| &\leq LM\\ &\leq \frac{2\pi \rho^{4/3}}{(a-\rho)(b-\rho)},\end{aligned}$$

which goes to zero as ρ goes to zero.
The integral over γ_+ is equal to

$$\int_{\gamma_+} \frac{\sqrt[3]{z}}{(z+a)(z+b)}\,\mathrm{d}z = \int_\rho^R \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,\mathrm{d}x$$

Finally, for the integral over γ_- we use the same parametrisation

$$z = x \qquad \text{where} \qquad x \in [\rho, R]$$

but with a different branch of the cube root

$$\sqrt[3]{z} = e^{2\pi i/3}\sqrt[3]{x}.$$

This traverses γ_- in the wrong direction, so we flip the sign.

$$\int_{\gamma_-} \frac{\sqrt[3]{z}}{(z+a)(z+b)}\,\mathrm{d}z = -e^{2\pi i/3}\int_\rho^R \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,\mathrm{d}x$$

Letting ρ go to zero and R go to infinity we get:

$$(1-e^{2\pi i/3})I = -2\pi i e^{\pi i/3}\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}.$$

Solving for I gives

$$\begin{aligned}
\int_0^\infty \frac{\sqrt[3]{x}\,\mathrm{d}x}{(x+a)(x+b)} &= I \\
&= -2\pi i\frac{e^{\pi i/3}}{1-e^{2\pi i/3}}\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \\
&= \pi\frac{2i}{e^{\pi i/3}-e^{-\pi i/3}}\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \\
&= \pi\frac{1}{\sin\pi/3}\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \\
&= \frac{2\pi}{\sqrt{3}}\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}.
\end{aligned}$$

5. Let

$$f(z) = \frac{(\log z)^2}{z^2+1}.$$

We use the branch of the logarithm and the indented contour of 2 (i). $f(z)$ has isolated singularities at $\pm i$ which are both simple poles but only the singularity at i belongs to U:

$$\begin{aligned}
\mathrm{Res}_i\, f(z) &= \lim_{z\to i}\frac{(\log z)^2}{2z} \\
&= \frac{(\pi i/2)^2}{2i} \\
&= \frac{\pi^2 i}{8}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}\int_\gamma \frac{(\log z)^2}{z^2+1}\,\mathrm{d}z &= 2\pi i \operatorname{Res}_i f(z)\\ &= 2\pi i \frac{\pi^2 i}{8}\\ &= -\frac{\pi^3}{4}.\end{aligned}$$

Next we show the integrals over γ_2 and γ_0 go to zero. As usual we have to estimate the largest value of $|f(z)|$. Over γ_2 we have

$$\begin{aligned}|f(z)| &= \frac{|\log z|^2}{|z^2+1|}\\ &\leq \frac{(\ln R + 2\pi)^2}{R^2-1}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{\gamma_2} \frac{(\log z)^2}{z^2+1}\,\mathrm{d}z\right| &\leq LM\\ &\leq \frac{\pi R(\ln R + 2\pi)^2}{R^2-1},\end{aligned}$$

which goes to zero as R goes to infinity. Over γ_0 we have

$$\begin{aligned}|f(z)| &= \frac{|\log z|^2}{|z^2+1|}\\ &\leq \frac{(2\pi - \ln\rho)^2}{1-\rho^2}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{\gamma_2} \frac{(\log z)^2}{z^2+1}\,\mathrm{d}z\right| &\leq LM\\ &\leq \frac{\pi\rho(2\pi - \ln\rho)^2}{1-\rho^2},\end{aligned}$$

which goes to zero as ρ goes to zero, since $\rho(\ln\rho)^2$ goes to zero. The integral over γ_+ is equal to

$$\int_{\gamma_+} \frac{(\log z)^2}{z^2+1}\,\mathrm{d}z = \int_\rho^R \frac{(\ln x)^2}{x^2+1}\,\mathrm{d}x.$$

Finally, for the integral over γ_- we use the parametrisation

$$z = -x \qquad \text{where} \qquad x \in [\rho, R].$$

In this case

$$\log z = \ln x + \pi i.$$

This traverses γ_- in the wrong direction, so we flip the sign.

$$\begin{aligned}
\int_{\gamma_-} \frac{(\log z)^2}{z^2+1}\,\mathrm{d}z &= \int_\rho^R \frac{(\ln x + \pi i)^2}{x^2+1}\,\mathrm{d}x \\
&= \int_\rho^R \frac{(\ln x)^2}{x^2+1}\,\mathrm{d}x + 2\pi i \int_\rho^R \frac{\ln x}{x^2+1}\,\mathrm{d}x - \pi^2 \int_\rho^R \frac{1}{x^2+1}\,\mathrm{d}x.
\end{aligned}$$

Letting ρ go to zero and R go to infinity we get:

$$2I = -\frac{\pi^3}{4} - 2\pi i \int_0^\infty \frac{\ln x}{x^2+1}\,\mathrm{d}x + \pi^2 \int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x.$$

We saw that

$$\int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2}$$

in lecture 2. If we take the imaginary part of both sides, we see that

$$\int_0^\infty \frac{\ln x}{x^2+1}\,\mathrm{d}x = 0.$$

Taking the real parts gives

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,\mathrm{d}x = \frac{\pi^3}{8}.$$