MODEL ANSWERS TO THE SECOND HOMEWORK

-1. $f(z)$ is differentiable wherever $p(z)$ and $q(z)$ are holomorphic and $q(z)$ is non-zero. In particular $f(z)$ is holomorphic in a punctured neighbourhood of a, so that $f(z)$ has an isolated singularity at a.
As

$$\frac{1}{f(z)} = \frac{q(z)}{p(z)}$$

has a simple zero at a it follows that $f(z)$ has a simple pole.
We calculate the residue there. There are two very similar ways to proceed:

$$\begin{aligned}\operatorname{Res}_a f(z) &= \lim_{z\to a} \frac{(z-a)p(z)}{q(z)} \\ &= \lim_{z\to a} \frac{p(z) + (z-a)p'(z)}{q'(z)} \\ &= \frac{p(a)}{q'(a)}.\end{aligned}$$

To get from the first line to the second line we used L'Hôpital's rule. Or we could proceed a little bit more directly:

$$\begin{aligned}\operatorname{Res}_a f(z) &= \lim_{z\to a} \frac{(z-a)p(z)}{q(z)} \\ &= \lim_{z\to a} \frac{p(z)}{\frac{q(z)}{z-a}} \\ &= \lim_{z\to a} \frac{p(z)}{\frac{q(z)-q(a)}{z-a}} \\ &= \frac{p(a)}{q'(a)}.\end{aligned}$$

0. We use the parameterisation

$$z = Re^{i\theta} \qquad \text{where} \qquad \theta \in [0, \pi].$$

In this case

$$|\mathrm{d}z| = R\mathrm{d}\theta.$$

On the other hand,

$$\begin{aligned}|e^{iz}| &= |e^{iRae^{i\theta}}| \\ &= |e^{iRa(\cos\theta + i\sin\theta)}| \\ &= |e^{iRa\cos\theta - aR\sin\theta}| \\ &= |e^{aiR\cos\theta}||e^{-aR\sin\theta}| \\ &= e^{-aR\sin\theta}.\end{aligned}$$

So we are reduced to showing that

$$\int_0^\pi e^{-aR\sin\theta}\,\mathrm{d}\theta < \frac{\pi}{Ra},$$

which we proved on the way to proving Jordan's Lemma.
1. Note first that

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,\mathrm{d}x,$$

is not absolutely convergent. If we replace $\sin ax$ by its absolute value, or what comes to pretty much the same thing, ignore $\sin ax$, the integrand becomes

$$\frac{x^3}{x^4+4}$$

which looks like $1/x$ for x large, whose integral diverges.
We proceed as usual but we will need to use the Cauchy principal value. We integrate around the usual contour and we let

$$f(z) = \frac{z^3 e^{iaz}}{z^4+4}.$$

This has poles at the roots of z^4+4. This has four roots and as before the two in the upper half plane are $e^{\pi i/4}$ and $e^{3\pi i/4}$. So the singularities of $f(z)$ in the upper half plane are

$$\sqrt{2}e^{\pi i/4} \qquad \text{and} \qquad \sqrt{2}e^{3\pi i/4}.$$

We compute the residues. Both are simple poles. We have

$$\begin{aligned}\operatorname{Res}_{\sqrt{2}e^{\pi i/4}} f(z) &= \lim_{z\to\sqrt{2}e^{\pi i/4}} \frac{z^3 e^{iaz}}{4z^3} \\ &= \lim_{z\to\sqrt{2}e^{\pi i/4}} \frac{e^{iaz}}{4} \\ &= \frac{1}{4}e^{ia\sqrt{2}e^{\pi i/4}} \\ &= \frac{1}{4}e^{-a+ia}.\end{aligned}$$

Similarly

$$\begin{aligned}
\operatorname{Res}_{\sqrt{2}e^{3\pi i/4}} f(z) &= \lim_{z\to\sqrt{2}e^{3\pi i/4}} \frac{z^3e^{iaz}}{4z^3} \\
&= \lim_{z\to\sqrt{2}e^{3\pi i/4}} \frac{e^{iaz}}{4} \\
&= \frac{1}{4}e^{ia\sqrt{2}e^{3\pi i/4}} \\
&= \frac{1}{4}e^{-a-ia},
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{z^3e^{iaz}}{z^4+4}\,\mathrm{d}z &= 2\pi i\left(\operatorname{Res}_{\sqrt{2}e^{\pi i/4}} + \operatorname{Res}_{\sqrt{2}e^{3\pi i/4}}\right) \\
&= \frac{\pi i}{2}\left(e^{-a+ia} + e^{-a-ia}\right) \\
&= \frac{e^{-a}\pi i}{2}\left(e^{ia} + e^{-ia}\right) \\
&= e^{-a}\pi i\cos a.
\end{aligned}$$

We now estimate the integral over γ_2. Once again this is more delicate than usual and we need to use to Jordan's Lemma:

$$\begin{aligned}
\left|\int_\gamma \frac{z^3e^{iaz}}{z^4+4}\,\mathrm{d}z\right| &\le \int_\gamma \frac{|z^3e^{iaz}|}{|z^4+4|}\,|\mathrm{d}z| \\
&= \int_\gamma \frac{|R^3e^{iaz}|}{R^4-4}\,|\mathrm{d}z| \\
&= \frac{R^3}{R^4-4}\int_\gamma |e^{iaz}|\,|\mathrm{d}z| \\
&< \frac{\pi R^3}{R^4-4},
\end{aligned}$$

which goes to zero, as R goes to infinity.
It follows that the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{x^3e^{iax}}{x^4+4}\,\mathrm{d}x,$$

is

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x^3e^{iax}}{x^4+4}\,\mathrm{d}x = e^{-a}\pi i\cos a.$$

Taking imaginary parts we get that the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4} \, \mathrm{d}x,$$

is

$$e^{-a}\pi \cos a.$$

As the integrand

$$\frac{x^3 \sin ax}{x^4+4}$$

is even, it follows that the improper integral converges to the Cauchy principal value:

$$\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4} \, \mathrm{d}x = \pi e^{-a} \cos a.$$

2. We integrate around the usual contour and we let

$$f(z) = \frac{ze^{iz}}{z^2+2z+2}.$$

This has poles at the roots of

$$z^2 + 2z + 2 = (z+1)^2 + 1.$$

It follows that the roots are

$$-1 \pm i.$$

So the only singularity of $f(z)$ in the upper half plane is $1+i$. As this is a simple pole, we have

$$\begin{aligned} \operatorname{Res}_{1+i} f(z) &= \lim_{z \to 1+i} \frac{(z-1-i)ze^{iz}}{z^2+2z+2} \\ &= \lim_{z \to 1+i} \frac{ze^{iz}}{z-1+i} \\ &= \frac{(1-i)e^{-1+i}}{2}. \end{aligned}$$

The residue theorem implies that

$$\begin{aligned} \int_{\gamma} \frac{ze^{iaz}}{z^2+2z+2} \, \mathrm{d}z &= 2\pi i \operatorname{Res}_{1+i} f(z) \\ &= \pi(1+i)e^{-1+i} \\ &= \frac{\pi}{e}(1+i)e^{i} \\ &= \frac{\pi}{e}(1+i)(\cos 1 + i \sin 1). \end{aligned}$$

We now estimate the integral over γ_2. Once again this is more delicate than usual and we need to use to Jordan's Lemma:

$$\begin{aligned}
\left|\int_\gamma \frac{ze^{iz}}{z^2+2z+2}\mathrm{d}z\right| &\leq \int_\gamma \frac{|ze^{iaz}|}{|z^2+2z+2|}\,|\mathrm{d}z| \\
&= \int_\gamma \frac{|Re^{iaz}|}{R^2-R-2}\,|\mathrm{d}z| \\
&= \frac{R}{R^2-R-2}\int_\gamma |e^{iaz}|\,|\mathrm{d}z| \\
&< \frac{\pi R}{R^2-R-2},
\end{aligned}$$

which goes to zero, as R goes to infinity.
It follows that the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+2x+2}\,\mathrm{d}x,$$

is

$$\frac{\pi}{e}(1+i)(\cos 1 + i\sin 1).$$

Taking imaginary parts we get that the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{x\sin x\mathrm{d}x}{x^2+2x+2}$$

is

$$\frac{\pi}{e}(\cos 1 + \sin 1).$$

3. (a) We integrate

$$f(z) = e^{iz^2}$$

around the closed contour

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3$$

where γ_1 goes from 0 to R along the real axis, γ_2 goes along the arc of the circle of radius R centred at the origin from R to $Re^{i\pi/4}$ and γ_3 goes along the straight line connecting $Re^{i\pi/4}$ to the origin.
$f(z)$ is entire. Cauchy's theorem implies that

$$\int_\gamma e^{iz^2}\,\mathrm{d}z = 0.$$

For the integral along γ_1 we have

$$\begin{aligned}\int_{\gamma_1} e^{iz^2}\,\mathrm{d}z &= \int_0^R e^{ix^2}\,\mathrm{d}x \\ &= \int_0^R \cos(x^2) + i\sin(x^2)\,\mathrm{d}x \\ &= \int_0^R \cos(x^2)\,\mathrm{d}x + i\int_0^R \sin(x^2)\,\mathrm{d}x.\end{aligned}$$

For the integral along $-\gamma_3$ we use the parameterisation

$$z = e^{i\pi/4}t \qquad \text{where} \qquad t \in [0, R].$$

$$\begin{aligned}-\int_{\gamma_3} e^{iz^2}\,\mathrm{d}z &= e^{i\pi/4}\int_0^R e^{i(e^{i\pi/4}t)^2}\,\mathrm{d}t \\ &= e^{i\pi/4}\int_0^R e^{i(e^{i\pi}t^2}\,\mathrm{d}t \\ &= e^{i\pi/4}\int_0^R e^{-t^2}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2}}\int_0^R e^{-t^2}\,\mathrm{d}t + \frac{i}{\sqrt{2}}\int_0^R e^{-t^2}\,\mathrm{d}t.\end{aligned}$$

Taking real and imaginary parts of the equation

$$\int_{\gamma_1} e^{iz^2}\,\mathrm{d}z = -\int_{\gamma_3} e^{iz^2}\,\mathrm{d}z - \int_{\gamma_2} e^{iz^2}\,\mathrm{d}z$$

gives

$$\begin{aligned}\int_0^R \cos(x^2)\,\mathrm{d}x &= \frac{1}{\sqrt{2}}\int_0^R e^{-t^2}\,\mathrm{d}t - \mathrm{Re}\int_{\gamma_2} e^{iz^2}\,\mathrm{d}z \\ \int_0^R \sin(x^2)\,\mathrm{d}x &= \frac{1}{\sqrt{2}}\int_0^R e^{-t^2}\,\mathrm{d}t - \mathrm{Im}\int_{\gamma_2} e^{iz^2}\,\mathrm{d}z.\end{aligned}$$

(b) Now we estimate the integral along γ_2. We have

$$\left|\int_{\gamma_2} e^{iz^2}\,\mathrm{d}z\right| \leq \int_{\gamma_2} |e^{iz^2}|\,|\mathrm{d}z|$$

For the integral on the RHS we use the parameterisation

$$z = Re^{i\theta} \qquad \text{where} \qquad \theta \in [0, \pi/4].$$

In this case

$$|\mathrm{d}z| = R\mathrm{d}\theta.$$

On the other hand,

$$\begin{aligned}|e^{iz^2}| &= |e^{iR^2e^{2i\theta}}|\\ &= |e^{iR^2(\cos 2\theta + i\sin 2\theta)}|\\ &= |e^{iR^2\cos 2\theta - R^2\sin 2\theta}|\\ &= |e^{iR^2\cos 2\theta}||e^{-R^2\sin 2\theta}|\\ &= e^{-R^2\sin 2\theta}.\end{aligned}$$

Making the change of variable $\phi = 2\theta$ are reduced to bounding

$$\begin{aligned}\int_{\gamma_2} |e^{iz^2}|\,|\mathrm{d}z| &= \int_0^{\pi/4} e^{-R^2\sin 2\theta}\,\mathrm{d}\theta\\ &= \int_0^{\pi/2} e^{-R^2\sin\phi}\,\mathrm{d}\phi\\ &= \frac{1}{2}\int_0^{\pi} e^{-R^2\sin\phi}\,\mathrm{d}\phi\\ &< \frac{\pi}{2R^2},\end{aligned}$$

which goes to zero as R goes to infinity.
(c) Letting R go to ∞ we get

$$\begin{aligned}\int_0^{\infty}\cos(x^2)\,\mathrm{d}x &= \frac{\sqrt{\pi}}{2\sqrt{2}}\\ \int_0^{\infty}\sin(x^2)\,\mathrm{d}x &= \frac{\sqrt{\pi}}{2\sqrt{2}}.\end{aligned}$$

4. We integrate around the unit circle and we use the parameterisation

$$z = \gamma(\theta) = e^{i\theta} \qquad \text{so that} \qquad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}.$$

Note that

$$\begin{aligned}\sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}\\ &= \frac{z - \frac{1}{z}}{2i}.\end{aligned}$$

We get

$$\begin{aligned}\int_0^{2\pi}\frac{\mathrm{d}\theta}{5 + 4\sin\theta} &= \oint_{|z|=1}\frac{1}{iz(5 + 4/2i(z - 1/z))}\,\mathrm{d}z\\ &= \oint_{|z|=1}\frac{1}{5iz + 2z^2 - 2}\,\mathrm{d}z.\end{aligned}$$

The integrand

$$\frac{1}{2z^2 + 5iz - 2}$$

has isolated singularities at the roots

$$2z^2 + 5iz - 2.$$

As this is a quadratic polynomial, we can apply the quadratic formula to find the roots:

$$\begin{aligned}\frac{-5i \pm \sqrt{-25+16}}{4} &= \frac{-5i \pm \sqrt{-9}}{4} \\ &= i\frac{-5 \pm 3}{4}.\end{aligned}$$

$-2i$ does not belong to the open unit disk Δ but $-i/2$ does belong to the open unit disk Δ. The singularities of $f(z)$ are simple, so that

$$\begin{aligned}\operatorname{Res}_{-i/2} f(z) &= \lim_{z \to -i/2} \frac{z + i/2}{2z^2 + 5iz - 2} \\ &= \lim_{z \to -i/2} \frac{1}{4z + 5i} \\ &= \frac{1}{3i}.\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 4\sin\theta} &= \oint_{|z|=1} \frac{1}{5iz + 2z^2 - 2}\,\mathrm{d}z \\ &= 2\pi i \frac{1}{3i} \\ &= \frac{2\pi}{3}.\end{aligned}$$

We can check this using the results in lecture 5, example 5.1. First of all sin and cos are related by a phase shift:

$$\cos\theta = \sin(\theta + \pi/2).$$

Since we are integrating sin over 2π it follows that

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\sin\theta} &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\sin(\theta+\pi/2)} \\ &= \int_0^{2\pi} \frac{\mathrm{d}\theta}{5+4\cos\theta} \\ &= \frac{1}{4}\int_0^{2\pi} \frac{\mathrm{d}\theta}{5/4+\cos\theta} \\ &= \frac{1}{4}\frac{2\pi}{\sqrt{(5/4)^2-1}} \\ &= \frac{1}{4}\frac{2\pi}{3/4} \\ &= \frac{2\pi}{3}.\end{aligned}$$

5.

$$\int_0^{2\pi} \frac{\cos^2 3\theta\,\mathrm{d}\theta}{5-4\cos 2\theta}.$$

We integrate around the unit circle and we use the parameterisation

$$z=\gamma(\theta)=e^{i\theta} \qquad \text{so that} \qquad \mathrm{d}\theta=\frac{\mathrm{d}z}{iz}.$$

Note that

$$\begin{aligned}\cos m\theta &= \frac{e^{mi\theta}+e^{-mi\theta}}{2} \\ &= \frac{z^m+\frac{1}{z^m}}{2}.\end{aligned}$$

We get

$$\begin{aligned}\int_0^{2\pi} \frac{\cos^2 3\theta\,\mathrm{d}\theta}{5-4\cos 2\theta} &= \oint_{|z|=1} \frac{1/4(z^3+1/z^3)^2}{iz(5-4/2(z^2+1/z^2))}\,\mathrm{d}z \\ &= \frac{i}{4}\oint_{|z|=1} \frac{(z^6+1)^2}{z^5(2z^4-5z^2+2)}\,\mathrm{d}z.\end{aligned}$$

The integrand

$$\frac{(z^6+1)^2}{z^5(2z^4-5z^2+2)}$$

has isolated singularities at the roots of

$$z^5(2z^4-5z^2+2).$$

Either $z = 0$ or we have a root of $2z^4 - 5z^2 + 2$. As this is a quadratic polynomial in z^2, we can apply the quadratic formula to find the roots:

$$\begin{aligned} \frac{5 \pm \sqrt{25-16}}{4} &= \frac{5 \pm \sqrt{9}}{4} \\ &= \frac{5 \pm 3}{4}. \end{aligned}$$

If z belongs to the open unit disk then so does z^2. 2 does not belong to the open unit disk Δ but $1/2$ does belong to the open unit disk Δ. Thus $f(z)$ has three singularities inside the unit disk, one at 0 and two at $\pm\frac{1}{\sqrt{2}}$.

The singularities of $f(z)$ at $\pm\frac{1}{\sqrt{2}}$ are simple so that

$$\begin{aligned} \text{Res}_{1/\sqrt{2}} f(z) &= \lim_{z \to 1/\sqrt{2}} \frac{(z^6+1)^2}{5z^4(2z^4 - 5z^2 + 2) + z^5(8z^3 - 10z)} \\ &= \lim_{z \to 1/\sqrt{2}} \frac{(z^6+1)^2}{2z^6(4z^2 - 5)} \\ &= \frac{((1/2)^3 + 1)^2}{1/4(2-5)} \\ &= -\frac{27}{16}. \end{aligned}$$

and

$$\begin{aligned} \text{Res}_{-1/\sqrt{2}} f(z) &= \lim_{z \to -1/\sqrt{2}} \frac{(z^6+1)^2}{5z^4(2z^4 - 5z^2 + 2) + z^5(8z^3 - 10z)} \\ &= \lim_{z \to -1/\sqrt{2}} \frac{(z^6+1)^2}{2z^6(4z^2 - 5)} \\ &= \frac{((1/2)^3 + 1)^2}{1/4(2-5)} \\ &= -\frac{27}{16}. \end{aligned}$$

The pole at 0 is a pole of order 5. So we want the coefficient of z^4 in the power series expansion of

$$\frac{(z^6+1)^2}{2z^4 - 5z^2 + 2}.$$

This is the same as the coefficient of z^4 in the power series expansion of

$$\frac{1}{2z^4 - 5z^2 + 2} = \frac{1}{2}\frac{1}{1 - 5/2z^2 + z^4}.$$

This coefficient is

$$\frac{1}{2}\left(-1+\frac{25}{4}\right)=\frac{21}{8}.$$

The residue theorem implies that

$$\begin{aligned}\int_0^{2\pi}\frac{\cos^2 3\theta\,\mathrm{d}\theta}{5-4\cos 2\theta} &= \frac{i}{4}\oint_{|z|=1}\frac{(z^6+1)^2}{z^5(2z^4-5z^2+2)}\,\mathrm{d}z \\ &= 2\pi i\frac{i}{4}\left(\operatorname{Res}_{1/\sqrt{2}} f(z)+\operatorname{Res}_{-1/\sqrt{2}} f(z)+\operatorname{Res}_0 f(z)\right) \\ &= 2\pi i\frac{i}{4}\left(-\frac{27}{8}+\frac{21}{8}\right) \\ &= \frac{3\pi}{8}.\end{aligned}$$

6. We integrate around the unit circle and we use the parameterisation

$$z=\gamma(\theta)=e^{i\theta} \qquad \text{so that} \qquad \mathrm{d}\theta=\frac{\mathrm{d}z}{iz}.$$

We get

$$\begin{aligned}\int_0^{\pi}\frac{\mathrm{d}\theta}{(a+\cos\theta)^2} &= \oint_{|z|=1}\frac{1}{iz(a+1/2(z+1/z))^2}\,\mathrm{d}z \\ &= \frac{4}{i}\oint_{|z|=1}\frac{z}{(2az+z^2+1)^2}\,\mathrm{d}z \\ &= \frac{4}{i}\oint_{|z|=1}\frac{z}{(2az+z^2+1)^2}\,\mathrm{d}z\end{aligned}$$

The integrand

$$\frac{z}{(2az+z^2+1)^2}$$

has isolated singularities at the roots of

$$z^2+2az+1=(z+a)^2+(1-a^2).$$

Therefore the singularities are at

$$-a\pm\sqrt{a^2-1}.$$

The negative square root surely does not belong to Δ but the positive one does:

$$\alpha=-a+\sqrt{a^2-1}\in\Delta.$$

As α is a double pole of $f(z)$ we have

$$\begin{aligned}
\operatorname{Res}_\alpha f(z) &= \lim_{z\to\alpha} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(z-\alpha)^2 z}{(z^2+2az+1)^2}\right) \\
&= \lim_{z\to\alpha} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z}{(z+a+\sqrt{a^2-1})^2}\right) \\
&= \lim_{z\to\alpha} \frac{(z+a+\sqrt{a^2-1})^2 - 2z(z+a+\sqrt{a^2-1})}{(z+a+\sqrt{a^2-1})^4} \\
&= \lim_{z\to\alpha} \frac{z+a+\sqrt{a^2-1}-2z}{(z+a+\sqrt{a^2-1})^3} \\
&= \lim_{z\to\alpha} \frac{a+\sqrt{a^2-1}-z}{(z+a+\sqrt{a^2-1})^3} \\
&= \frac{2a}{(2\sqrt{a^2-1})^3} \\
&= \frac{a}{4(\sqrt{a^2-1})^3}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_0^\pi \frac{\mathrm{d}\theta}{(a+\cos\theta)^2} &= \frac{4}{i}\oint_{|z|=1} \frac{z}{(2az+z^2+1)^2}\,\mathrm{d}z \\
&= 2\pi i \frac{4}{i}\frac{a}{4(\sqrt{a^2-1})^3} \\
&= \frac{2\pi a}{(\sqrt{a^2-1})^3}.
\end{aligned}$$

7. As the integrand

$$\sin^{2n}\theta = (\sin^n\theta)^2$$

is a square, it is even. Therefore

$$\begin{aligned}
\int_0^\pi \sin^{2n}\theta\,\mathrm{d}\theta &= \frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta\,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\,\mathrm{d}\theta.
\end{aligned}$$

To calculate the last integral, we integrate around the unit circle and we use the parameterisation

$$z = \gamma(\theta) = e^{i\theta} \qquad \text{so that} \qquad \mathrm{d}z = \frac{\mathrm{d}\theta}{iz}.$$

We have

$$\int_0^{2\pi} \sin^{2n}\theta \, \mathrm{d}\theta = \oint_{|z|=1} \frac{1}{iz(2i)^{2n}} \left(z - \frac{1}{z}\right)^{2n} \mathrm{d}z$$
$$= \frac{1}{2^{2n} i (-1)^n} \oint_{|z|=1} \frac{(z^2-1)^{2n}}{z^{2n+1}} \mathrm{d}z$$

The integrand

$$f(z) = \frac{(z^2-1)^{2n}}{z^{2n+1}}$$

has a pole of order $2n+1$ at $0 \in \Delta$. To compute the residue there, probably the most efficient way to proceed is to use the binomial theorem to expand the numerator. Since we want the coefficient of $1/z$ for the Laurent expansion of $f(z)$, we want the coefficient of z^{2n} in the binomial expansion of the numerator $(z^2-1)^{2n}$. This is is the same as the coefficient of x^n in the binomial expansion of $(x-1)^{2n}$, which is

$$(-1)^n \binom{2n}{n}.$$

The residue theorem therefore implies that

$$\int_0^{\pi} \sin^{2n}\theta \, \mathrm{d}\theta = \frac{1}{2} \int_0^{2\pi} \sin^{2n}\theta \, \mathrm{d}\theta$$
$$= \frac{1}{2} \frac{1}{2^{2n} i (-1)^n} \oint_{|z|=1} \frac{(z^2-1)^{2n}}{z^{2n+1}} \mathrm{d}z$$
$$= \frac{1}{2} \frac{1}{2^{2n} i (-1)^n} 2\pi i (-1)^n \binom{2n}{n}$$
$$= \frac{1}{2^{2n}} \pi \binom{2n}{n}$$
$$= \frac{(2n)!}{2^{2n}(n!)^2} \pi.$$

Challenge Problems: (Just for fun)

8. Calculate

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4 + ax^2 + b^2} \qquad \text{where} \qquad a > 0, b > 0, a^2 \geq 4b^2.$$