9. Inserting the logarithm

Example 9.1. *Calculate*

$$I = \int_0^\infty \frac{\mathrm{d}x}{x^3 + 1}.$$

At first sight this integral seems straightforward, just integrate

$$\frac{1}{z^3 + 1}$$

over the standard contour. The problem is that if we do this then we get the value of

$$\int_{-\infty}^\infty \frac{\mathrm{d}x}{x^3 + 1}$$

and there is no obvious way to go from the value of this integral to the value of the integral we are after, since $x^3 + 1$ is neither odd nor even.

We already saw one fix in a homework problem, integrate along an arc instead of a semicircle. If we go along the line from $e^{2\pi i/3}R$ to 0 then we can exploit the fact that

$$\frac{1}{(e^{2\pi i/3}t)^3 + 1} = \frac{1}{t^3 + 1}.$$

Here is another way to proceed which is more versatile. It looks as though using a keyhole contour in Lecture 8 might work, since we only integrate along the interval $[\rho, R]$. The problem with using the keyhole contour is that there is no ambiguity in the definition of

$$\frac{1}{z^3 + 1}$$

so that when we integrate over $\gamma_- + \gamma_+$ the integrals cancel.

To engineer an integral that does not cancel we integrate

$$f(z) = \frac{\log z}{z^3 + 1}$$

instead. We use the same branch of the logarithm as in Lecture 8. We cut along the positive real axis:

$$\log z = \ln|z| + i \arg z \qquad \text{where} \qquad \arg z \in (0, 2\pi)$$

so that $\log z$ is holomorphic on

$$V = \mathbb{C} \setminus \{\, x \mid x \geq 0 \,\}.$$

$f(z)$ has isolated singularities at the cube roots of -1,

$$e^{\pi i/3}, \qquad e^{3\pi i/3} \qquad \text{and} \qquad e^{5\pi i/3}.$$

These are all simple singularities. As they all have modulus one the logarithm is purely imaginary at these points. We compute the residues:

$$\begin{aligned}\operatorname{Res}_{e^{\pi i/3}} f(z) &= \lim_{z\to e^{\pi i/3}} \frac{\log z}{3z^2}\\ &= \frac{\pi i/3}{3e^{2\pi i/3}}\\ &= \frac{\pi i e^{-2\pi i/3}}{9}.\end{aligned}$$

We also have

$$\begin{aligned}\operatorname{Res}_{e^{3\pi i/3}} f(z) &= \lim_{z\to e^{\pi i}} \frac{\log z}{3z^2}\\ &= \frac{\pi i}{3e^{2\pi i}}\\ &= \frac{\pi i}{3},\end{aligned}$$

and

$$\begin{aligned}\operatorname{Res}_{e^{5\pi i/3}} f(z) &= \lim_{z\to e^{5\pi i/3}} \frac{\log z}{3z^2}\\ &= \frac{5\pi i/3}{3e^{10\pi i/3}}\\ &= \frac{5\pi i e^{-4\pi i/3}}{9}.\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}\int_\gamma \frac{\log z}{z^3+1}\,\mathrm{d}z &= 2\pi i\left(\operatorname{Res}_{e^{\pi i/3}} f(z) + \operatorname{Res}_{e^{3\pi i/3}} f(z) + \operatorname{Res}_{e^{5\pi i/3}} f(z)\right)\\ &= 2\pi i\frac{\pi i}{9}\left(e^{-2\pi i/3} + 3 + 5e^{-4\pi i/3}\right)\\ &= 2\pi i\frac{\pi i}{9}\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2} + 3 - \frac{5}{2} + \frac{5\sqrt{3}i}{2}\right)\\ &= -2\pi i\frac{2\pi}{3\sqrt{3}}.\end{aligned}$$

Next we show the integrals over γ_2 and γ_0 go to zero. As usual we have to estimate the largest value of $|f(z)|$. Over γ_2 we have

$$\begin{aligned}|f(z)| &= \frac{|\log z|}{|z^3+1|}\\ &\le \frac{\ln R + 2\pi}{R^3-1}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\log z}{z^3+1}\,\mathrm{d}z\right| &\le LM \\ &\le \frac{2\pi R(\ln R + 2\pi)}{R^3 - 1},\end{aligned}$$

which goes to zero as R goes to infinity. Over γ_0 we have

$$\begin{aligned}|f(z)| &= \frac{|\log z|}{|z^3+1|} \\ &\le \frac{2\pi - \ln \rho}{1 - \rho^3}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_{\gamma_2} \frac{\log z}{z^3+1}\,\mathrm{d}z\right| &\le LM \\ &\le \frac{2\pi\rho(2\pi - \ln \rho)}{1 - \rho^3},\end{aligned}$$

which goes to zero as ρ goes to zero, since $\rho \ln \rho$ goes to zero.

The integral over γ_+ is equal to

$$\int_{\gamma_+} \frac{\log z}{z^3+1}\,\mathrm{d}z = \int_\rho^R \frac{\ln x}{x^3+1}\,\mathrm{d}x.$$

Finally, for the integral over γ_- we use the same parametrisation

$$z = x \qquad \text{where} \qquad x \in [\rho, R]$$

but with a different branch of the logarithm

$$\log z = \ln x + 2\pi i.$$

This traverses γ_- in the wrong direction, so we flip the sign.

$$\int_{\gamma_-} \frac{\log z}{z^3+1}\,\mathrm{d}z = -\int_\rho^R \frac{\ln x}{x^3+1}\,\mathrm{d}x - 2\pi i \int_\rho^R \frac{\mathrm{d}x}{x^3+1}.$$

Letting ρ go to zero and R go to infinity we get:

$$-2\pi i I = -2\pi i \frac{2\pi}{3\sqrt{3}}.$$

Solving for I gives

$$\int_0^\infty \frac{\mathrm{d}x}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$