8. The keyhole contour

Example 8.1. *Calculate*

$$I = \int_0^\infty \frac{x^a}{(x+1)^2}\,\mathrm{d}x \qquad \textit{where} \qquad a \in (-1, 1).$$

Clearly this example is very similar to the one in lecture 7. We proceed as before, we introduce

$$f(z) = \frac{z^a}{(z+1)^2}.$$

However now we choose to make a branch cut along the positive real line.

$$V = \mathbb{C} \setminus \{\, x \mid x \geq 0 \,\}.$$

We are going to make a choice of $\log z$ with a cut along the positive real axis:

$$\log z = \ln |z| + i \arg z \qquad \text{where} \qquad \arg z \in (0, 2\pi).$$

This makes $\log z$ a holomorphic function on V. From here, it is easy to define

$$z^a = e^{a \log z}.$$

This makes z^a a holomorphic function on V.

We are going to make an usual choice of contour. The contour will have four parts:

$$\gamma = \gamma_+ + \gamma_0 + \gamma_R + \gamma_-.$$

But now the problem is that we have to exclude 0 from the contour, since 0 is part of the cut. So we use the same indented path as in lecture 6, which has four pieces:

$$\gamma = \gamma_- + \gamma_0 + \gamma_+ + \gamma_2.$$

γ_+ goes from ρ to R and γ_2 goes from R to R all the way around a circle of radius R. γ_- goes from R back to ρ and γ_0 goes all the way around a circle of radius ρ, clockwise.

Strictly speaking the paths $\gamma_\pm$ make no sense. For a start one is the reverse of the other and secondly both are along the cut, where $f(z)$ is even defined. However we imagine that γ_+ is really above the cut and γ_- is really below the cut. This gives a different value to z^a on γ_+ and γ_-.

The function $f(z)$ has an isolated singularity at -1 which belongs inside the contour, as long as $R > 1$ and $\rho < 1$. We compute the residue at -1. This is a double pole:

$$\begin{aligned}
\operatorname{Res}_{-1} f(z) &= \lim_{z\to -1} \frac{\mathrm{d}}{\mathrm{d}z}\left(z^a\right) \\
&= \lim_{z\to -1} a z^{a-1} \\
&= a(-1)^{a-1} \\
&= -a(e^{i\pi})^a \\
&= -a e^{ai\pi}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{z^a}{(z+1)^2}\,\mathrm{d}z &= 2\pi i \operatorname{Res}_1 f(z) \\
&= -2\pi i a e^{ai\pi}.
\end{aligned}$$

We estimate the integral over γ_2. We estimate the maximum value M of $|f(z)|$ over γ_2:

$$\begin{aligned}
|f(z)| &= \left|\frac{z^a}{(z+1)^2}\right| \\
&= \frac{|z^a|}{|(z+1)^2|} \\
&\leq \frac{R^a}{(R-1)^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{z^a}{(z+1)^2}\,\mathrm{d}z\right| &\leq LM \\
&\leq \frac{2\pi R^{a+1}}{(R-1)^2},
\end{aligned}$$

which goes to zero as R goes to infinity, since $a+1<2$.

Now we compute what happens over γ_0 as ρ goes to zero. We estimate the maximum value M of $|f(z)|$ over γ_0:

$$\begin{aligned}
|f(z)| &= \left|\frac{z^a}{(z+1)^2}\right| \\
&= \frac{|z^a|}{|(z+1)^2|} \\
&\leq \frac{\rho^a}{(1-\rho)^2}.
\end{aligned}$$

It follows that

$$\left|\int_{\gamma_0} \frac{z^a}{(z+1)^2}\,\mathrm{d}z\right| \leq LM$$
$$\leq \frac{2\pi\rho^{a+1}}{(1-\rho)^2},$$

which goes to zero as ρ goes to zero, since $a+1>0$.

The integral over γ_+ is equal to

$$\int_{\gamma_+} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = \int_\rho^R \frac{x^a}{(x^2+1)^2}\,\mathrm{d}x$$

which goes to the value of the improper integral I we are trying to compute, as ρ goes to zero and R to infinity.

Finally, for the integral over γ_- we use the same parametrisation

$$z = x \qquad \text{where} \qquad x \in [\rho, R]$$

but a different branch of the logarithm

$$z^a = x^a e^{2\pi i a}.$$

This traverses γ_- in the wrong direction, so we flip the sign.

$$\int_{\gamma_-} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = -e^{2a\pi i} \int_\rho^R \frac{x^a}{(x^2+1)^2}\,\mathrm{d}x.$$

Letting ρ go to zero and R go to infinity we get:

$$(1-e^{2\pi i a})I = -2\pi i a e^{\pi i a}.$$

Solving for I gives

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,\mathrm{d}x = I$$
$$= -2\pi i a \frac{e^{a\pi i}}{1-e^{2a\pi i}}$$
$$= \pi a \frac{-2i}{e^{-a\pi i} - e^{a\pi i}}$$
$$= \pi a \frac{2i}{e^{a\pi i} - e^{-a\pi i}}$$
$$= \frac{\pi a}{\sin \pi a}.$$