7. Branch points

Example 7.1. *Calculate*

$$I = \int_0^\infty \frac{x^a}{(x^2+1)^2}\,\mathrm{d}x \qquad \textit{where} \qquad a \in (-1,3).$$

To solve this problem using contour integration we introduce

$$f(z) = \frac{z^a}{(z^2+1)^2}.$$

This immediately introduces a problem, that wasn't present for the improper integral. Namely z^a is not well-defined. For example, if we take $a = 1/2$ we are taking square roots.

To consistently choose a square root we need to cut the plane open along a straight line. Which straight line? The usual choice is the negative real axis.

Presumably we are going to integrate along the standard contour, so the negative real axis is a bad choice of branch cut, since a half (or better $1/(2+\pi)$) of the contour is along the negative real axis (actually we can make this work but this requires looking at quite a different contour). We don't want the branch cut in the upper half plane, so the best choice is to cut along the negative imaginary axis:

$$V = \mathbb{C} \setminus \{\, iy \,|\, y \leq 0 \,\}.$$

The best way to take roots is to use logarithms. So we are going to make a choice of $\log z$ with a cut along the negative imaginary axis:

$$\log z = \ln|z| + i \arg z \qquad \text{where} \qquad \arg z \in (-\pi/2, 3\pi/2).$$

This makes $\log z$ a holomorphic function on V. From here, it is easy to define

$$z^a = e^{a \log z}.$$

This makes z^a a holomorphic function on V.

But now the problem is that we have to exclude 0 from the contour, since 0 is part of the cut. So we use the same indented path as in lecture 6, which has four pieces:

$$\gamma = \gamma_- + \gamma_0 + \gamma_+ + \gamma_2.$$

The function $f(z)$ has isolated singularities at i which belongs to the upper half plane. It doesn't have a singularity at $-i$ since $f(z)$ is not defined along the whole negative imaginary axis. We suppose that $R > 1$ but $\rho < 1$, so that we capture i.

We compute the residue at i. This is a double pole:

$$\begin{aligned}
\operatorname{Res}_i f(z) &= \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{(z-i)^2 z^a}{(z^2+1)^2}\right) \\
&= \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^a}{(z+i)^2}\right) \\
&= \lim_{z\to i} \frac{az^{a-1}(z+i)^2 - 2z^a(z+i)}{(z+i)^4} \\
&= \lim_{z\to i} \frac{az^{a-1}(z+i) - 2z^a}{(z+i)^3} \\
&= \frac{ai^{a-1}2i - 2i^a}{(2i)^3} \\
&= \frac{(a-1)i^a}{-4i} \\
&= \frac{a-1}{4} i e^{a\pi i/2}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z &= 2\pi i \operatorname{Res}_i f(z) \\
&= \frac{1-a}{2}\pi e^{a\pi i/2}.
\end{aligned}$$

We estimate the integral over γ_2. We estimate the maximum value M of $|f(z)|$ over γ_2:

$$\begin{aligned}
|f(z)| &= \left|\frac{z^a}{(z^2+1)^2}\right| \\
&= \frac{|z^a|}{|(z^2+1)^2|} \\
&\le \frac{R^a}{(R^2-1)^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z\right| &\le LM \\
&\le \frac{\pi R^{a+1}}{(R^2-1)^2},
\end{aligned}$$

which goes to zero as R goes to infinity, since $a+1<4$.

Now we compute what happens over γ_0 as ρ goes to zero. We estimate the maximum value M of $|f(z)|$ over γ_0:

$$\begin{aligned}
|f(z)| &= \left|\frac{z^a}{(z^2+1)^2}\right| \\
&= \frac{|z^a|}{|(z^2+1)^2|} \\
&\leq \frac{\rho^a}{(1-\rho^2)^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left|\int_{\gamma_0} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z\right| &\leq LM \\
&\leq \frac{\pi\rho^{a+1}}{(1-\rho^2)^2},
\end{aligned}$$

which goes to zero as ρ goes to zero, since $a+1>0$. Note that the denominator approaches 1, so that it plays no role.

The integral over γ_+ is equal to

$$\int_{\gamma_+} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = \int_\rho^R \frac{x^a}{(x^2+1)^2}\,\mathrm{d}x$$

which goes to the value of the improper integral I we are trying to compute, as ρ goes to zero and R to infinity.

Finally, for the integral over γ_- we use the parametrisation

$$z = -x \qquad \text{where} \qquad x \in [\rho, R].$$

This traverses γ_- in the wrong direction, so we flip the sign.

$$\int_{\gamma_-} \frac{z^a}{(z^2+1)^2}\,\mathrm{d}z = e^{a\pi i}\int_\rho^R \frac{x^a}{(x^2+1)^2}\,\mathrm{d}x.$$

Note that exponential is simply $(-1)^a$.

Letting ρ go to zero and R go to infinity we get:

$$(1+e^{a\pi i})I = \frac{1-a}{2}\pi e^{a\pi i/2}.$$

Solving for I gives

$$\begin{aligned}\int_0^\infty \frac{x^a}{(x^2+1)^2}\,\mathrm{d}x &= I \\ &= \frac{1-a}{2}\pi\frac{e^{a\pi i/2}}{1+e^{a\pi i}} \\ &= \frac{1-a}{4}\pi\frac{2}{e^{-a\pi i/2}+e^{a\pi i/2}} \\ &= \frac{\pi(1-a)}{4\cos a\pi/2}.\end{aligned}$$