6. Indented Paths

Example 6.1. *Calculate*

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x.$$

This integral is called **Dirichlet's integral**.

We first observe that this integral is not absolutely convergent. Actually

$$\int_1^\infty \frac{\mathrm{d}x}{x}$$

diverges but so in fact does

$$\int_0^1 \frac{\mathrm{d}x}{x}.$$

Indeed,

$$\begin{aligned}
\int_1^\infty \frac{\mathrm{d}x}{x} &= \lim_{u\to\infty}\int_1^u \frac{\mathrm{d}x}{x}\\
&= \lim_{u\to\infty}\Big[\ln x\Big]_1^u\\
&= \lim_{u\to\infty}\ln u - \ln 1\\
&= \infty,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}x}{x} &= \lim_{l\to 0}\int_l^1 \frac{\mathrm{d}x}{x}\\
&= \lim_{l\to 0}\Big[\ln x\Big]_l^1\\
&= \lim_{l\to 0}\ln 1 - \ln l\\
&= -\infty.
\end{aligned}$$

We will need to use the Cauchy principal value both at infinity and at 0:

Definition 6.2. *Let $f(x)$ be a complex valued function on a finite interval*

$$f\colon (b,c)\longrightarrow \mathbb{C}.$$

We suppose that f is continuous except at $a\in(b,c)$.

The ***Cauchy principal value*** *is the limit (assuming it exists):*

$$\lim_{\epsilon\to 0}\left(\int_b^{a-\epsilon} f(x)\,\mathrm{d}x + \int_{a+\epsilon}^c f(x)\,\mathrm{d}x\right).$$

Here ϵ is a positive number decreasing to 0.

Example 6.3. *Consider*

$$\int_{-1}^{1} \frac{\mathrm{d}x}{x}.$$

The integrand is not continuous at 0.

This improper integral diverges but the Cauchy principal value exists. Indeed for the improper integral we have

$$\begin{aligned} \int_{-1}^{1} \frac{\mathrm{d}x}{x} &= \lim_{u\to 0, l\to 0} \int_{-1}^{u} \frac{\mathrm{d}x}{x} + \int_{l}^{1} \frac{\mathrm{d}x}{x} \\ &= \lim_{u\to 0, l\to 0} \ln u - \ln l. \end{aligned}$$

If we let u to zero first then we get $-\infty$ but if we let l go to zero first we get ∞. In fact we can get any limit we please, if we coordinate l and u. On the other hand, the Cauchy principal value is

$$\begin{aligned} \lim_{\epsilon\to 0} \left(\int_{-1}^{-\epsilon} \frac{\mathrm{d}x}{x} + \int_{\epsilon}^{1} \frac{\mathrm{d}x}{x} \right) &= \lim_{\epsilon\to 0} \ln -\epsilon - \ln \epsilon \\ &= \lim_{\epsilon\to 0} 0 \\ &= 0. \end{aligned}$$

Now let us go back to calculating the original integral. The function

$$\frac{\sin z}{z},$$

has an isolated singularity at the origin and the singularity there is removable. It follows that the integrand

$$\frac{\sin x}{x}$$

extends to a function on the whole real line. This function is even and so we compute

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} \, \mathrm{d}x$$

and divide by 2.

Proceeding as usual we consider

$$f(z) = \frac{e^{iz}}{z}.$$

It is tempting to integrate this over the standard contour. The problem is that $f(z)$ has an isolated singularity at 0 which is not removable, rather it is a simple pole.

Instead we integrate over a small perturbation of the standard path. We integrate from $-R$ to $-\rho$ along the real axis, γ_-; around a semicircle of radius ρ in the upper half plane from $-\rho$ to ρ, γ_0; from ρ to R along the real axis, γ_+; and then back along a semicircle of radius R, γ_2. As usual we let R go to infinity and we are going to let ρ go to zero.

The semicircle of radius ρ is the small indentation. Note that we traverse this semicircle clockwise, not anticlockwise. Let

$$\gamma = \gamma_- + \gamma_0 + \gamma_+ + \gamma_2$$

be the resulting closed contour.

Let U be the complement of the closed unit disc or radius ρ centred at the origin, inside the open unit disc of radius R centred at the origin in the upper half plane,

$$U = \{\, z \in \mathbb{H} \,|\, \rho < |z| < R \,\},$$

the intersection of an annulus with the upper half plane. Then the boundary of U is γ.

The only singularity of $f(z)$ is at the origin and this is neither a point of U nor a point of ∂U. Cauchy's theorem implies that

$$\int_\gamma \frac{e^{iz}}{z} \, \mathrm{d}z = 0.$$

We now compute each part of the integral over γ separately. The integral over γ_2 goes to zero, using Jordan's Lemma:

$$\begin{aligned} \left| \int_{\gamma_2} \frac{e^{iz}}{z} \, \mathrm{d}z \right| &\leq \int_{\gamma_2} \frac{|e^{iz}|}{|z|} \, |\mathrm{d}z| \\ &= \int_{\gamma_2} \frac{|e^{iz}|}{R} \, |\mathrm{d}z| \\ &= \frac{1}{R} \int_{\gamma_2} |e^{iz}||\mathrm{d}z| \\ &< \frac{\pi}{R}, \end{aligned}$$

which goes to zero, as R goes to infinity.

As we let R go to infinity and ρ go to zero then the integral over γ_- and γ_+ tends to the Cauchy principal value:

$$\lim_{R \to \infty, \rho \to 0} \left(\int_{-R}^{-\rho} \frac{e^{ix}}{x} \, \mathrm{d}x + \int_{\rho}^{R} \frac{e^{ix}}{x} \, \mathrm{d}x \right)$$

of the improper integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, \mathrm{d}x.$$

It remains to understand what happens around the semicircle of radius ρ, as ρ goes to zero. Now if we went all the way around the circle of radius ρ then we could compute this using the Residue Theorem.

In fact the integral around the semicircle approaches half of this, as ρ goes to zero:

Lemma 6.4. *Suppose that $f(z)$ has a simple pole at $a \in \mathbb{R}$ and γ_ρ is the semicircle of radius ρ centred at a in the upper half plane, traversed anticlockwise.*

Then

$$\lim_{\rho \to 0} \int_{\gamma_\rho} f(z)\,\mathrm{d}z = \pi i \operatorname{Res}_a f(z).$$

We defer the proof of (6.4) to the end and first show how to use it to finish the computation. We compute the residue at 0:

$$\begin{aligned} \operatorname{Res}_0 \frac{e^{iz}}{z} &= \lim_{z \to 0} e^{iz} \\ &= 1. \end{aligned}$$

(6.4) implies that

$$\lim_{\rho \to 0} \int_{\gamma_0} \frac{e^{iz}}{z}\,\mathrm{d}z = -\pi i.$$

Note the minus sign, since we traverse γ_0 clockwise.

Putting all of this together we see that the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,\mathrm{d}x$$

is πi. Taking the imaginary part, it follows that the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x$$

is π. Using the fact that $\sin x/x$ is even, it follows that the Cauchy principal value of

$$\int_{0}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x$$

is $\pi/2$. But this obviously agrees with the value of the improper integral

$$\int_{0}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

Proof. By assumption $f(z)$ has a Laurent expansion centred at a in a punctured neighbourhood of a, so that we may write

$$f(z) = \frac{a_{-1}}{z-a} + g(z),$$

where $g(z)$ is holomorphic at a.

If we parametrise γ_ρ in the obvious way,

$$\gamma_\rho(\theta) = \rho e^{i\theta} + a \qquad \text{where} \qquad \theta \in [0, \pi]$$

then

$$\mathrm{d}z = -\rho e^{i\theta}\,\mathrm{d}\theta$$

and so we get

$$\begin{aligned}\int_{\gamma_\rho} f(z)\,\mathrm{d}z &= \int_0^\pi i a_{-1}\,\mathrm{d}\theta + \int_{\gamma_\rho} g(z)\,\mathrm{d}z \\ &= \pi i a_{-1} + \int_{\gamma_\rho} g(z)\,\mathrm{d}z \\ &= \pi i \operatorname{Res}_a f(z) + \int_{\gamma_\rho} g(z)\,\mathrm{d}z.\end{aligned}$$

As $g(z)$ is holomorphic at a it is certainly continuous at a and so it is certainly bounded near a,

$$|g(z)| \leq M,$$

for some M. The semicircle of radius ρ has length $\pi\rho$ and so

$$\begin{aligned}\left|\int_{\gamma_\rho} g(z)\,\mathrm{d}z\right| &\leq LM \\ &= \pi\rho M,\end{aligned}$$

which goes to zero, as ρ goes to zero. □

Note that it isn't really important that a is a real number and there are similar results if one goes around the arc of any circle, we just pick up the corresponding proportion of the residue.