## 5. Trigonometric integrals

**Example 5.1.** *Calculate*

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} \qquad \textit{where} \qquad a > 1.$$

We are going to use contour integration to evaluate this integral. Clearly this integral has a different nature to the previous examples, since the range of integration is a finite interval.

However it does seems that we are using angles and that we are going around a circle. So let's try

$$U = \Delta,$$

the open unit disk. In this case the boundary is the unit circle. We use the standard parametrisation

$$z = \gamma(\theta) = e^{i\theta} \qquad \text{so that} \qquad \mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta = iz\,\mathrm{d}\theta.$$

It follows that

$$\mathrm{d}\theta = \frac{\mathrm{d}z}{iz}.$$

Now

$$\begin{aligned}\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}\\ &= \frac{z+\frac{1}{z}}{2}.\end{aligned}$$

This gives

$$\begin{aligned}\int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta} &= \oint_{|z|=1} \frac{\mathrm{d}z}{iz(a+\frac{z+\frac{1}{z}}{2})}\\ &= \frac{2}{i}\oint_{|z|=1} \frac{\mathrm{d}z}{z^2+2za+1}\end{aligned}$$

The integrand is

$$\frac{1}{z^2+2za+1}.$$

This has poles at the zeroes of

$$z^2+2az+1,$$

which are given by the quadratic formula,

$$\frac{-2a\pm\sqrt{4a^2-4}}{2} = -a\pm\sqrt{a^2-1}.$$

Now the negative square root doesn't belong to $\Delta$ and the positive square root

$$\alpha = -a + \sqrt{a^2 - 1}$$

does belong to $\Delta$. The positive square root is a simple pole. The residue is

$$\begin{aligned}
\operatorname{Res}_\alpha f(z) &= \lim_{z \to \alpha} \frac{z - \alpha}{z^2 + 2az + 1} \\
&= \lim_{z \to \alpha} \frac{1}{2z + 2a} \\
&= \frac{1}{2(\sqrt{a^2 - 1})}.
\end{aligned}$$

The residue theorem implies

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + \cos\theta} &= 2\pi i \frac{2}{i} \operatorname{Res}_\alpha f(z) \\
&= \frac{2\pi}{\sqrt{a^2 - 1}}.
\end{aligned}$$

**Example 5.2.** *Calculate*

$$\int_0^{\pi} \frac{\cos 2\theta \, \mathrm{d}\theta}{1 - 2a\cos\theta + a^2} \qquad \textit{where} \qquad a \in (-1, 1).$$

We first suppose that $a \neq 0$. Note that the graph of $\cos\theta$ is symmetric about the vertical line $\theta = \pi$ so that the integrand is symmetric about the same line.

So we calculate

$$\int_0^{2\pi} \frac{\cos 2\theta \, \mathrm{d}\theta}{1 - 2a\cos\theta + a^2}$$

and divide by 2.

We proceed as before, we integrate around the unit circle and we use the same parametrisation. We have

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos 2\theta \, \mathrm{d}\theta}{1 - 2a\cos\theta + a^2} &= \oint_\gamma \frac{z^2 + \frac{1}{z^2}}{2iz(1 - a(z + 1/z) + a^2)} \, \mathrm{d}z \\
&= \frac{i}{2} \oint_\gamma \frac{z^4 + 1}{z^2(az^2 + a - z - a^2 z)} \, \mathrm{d}z \\
&= \frac{i}{2} \oint_\gamma \frac{z^4 + 1}{z^2(z - a)(az - 1)} \, \mathrm{d}z
\end{aligned}$$

The integrand is

$$f(z) = \frac{z^4+1}{z^2(z-a)(az-1)}.$$

This has isolated singularities at 0, $a$ and $1/a$. Since $a \in (-1,1)$, on the first two belong to the unit disk.

The first singularity is a double pole:

$$\begin{aligned}
\operatorname{Res}_0 f(z) &= \lim_{z\to 0} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^4+1}{(z-a)(az-1)}\right) \\
&= \lim_{z\to 0} \frac{4z^3(z-a)(az-1)-(z^4+1)(2az-1-a^2)}{(z-a)^2(az-1)^2} \\
&= \frac{a^2+1}{a^2}.
\end{aligned}$$

The second is a simple pole

$$\begin{aligned}
\operatorname{Res}_a f(z) &= \lim_{z\to a} \frac{z^4+1}{z^2(az-1)} \\
&= \frac{a^4+1}{a^2(a^2-1)}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\oint_\gamma \frac{z^4+1}{z^2(z-a)(az-1)}\,\mathrm{d}z &= 2\pi i\left(\operatorname{Res}_0 f(z) + \operatorname{Res}_a f(z)\right) \\
&= 2\pi i\left(\frac{a^2+1}{a^2} + \frac{a^4+1}{a^2(a^2-1)}\right) \\
&= 2\pi i\left(\frac{a^4-1}{a^2(a^2-1)} + \frac{a^4+1}{a^2(a^2-1)}\right) \\
&= 2\pi i \frac{2a^2}{a^2-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_0^\pi \frac{\cos 2\theta\,\mathrm{d}\theta}{1-2a\cos\theta+a^2} &= \frac{i}{4}\cdot 2\pi i \frac{2a^2}{a^2-1} \\
&= \frac{\pi a^2}{1-a^2}.
\end{aligned}$$