4. Conditional convergence

Example 4.1. *Calculate*

$$\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^2+3}\,\mathrm{d}x.$$

It is not immediately clear that the integral above converges. Suppose we ignored the sine term to get

$$\int_{-\infty}^{\infty} \frac{x}{x^2+3}\,\mathrm{d}x.$$

This integral diverges. The integrand is

$$\frac{x}{x^2+3}$$

which when x gets large looks like $1/x$. But

$$\begin{aligned}\int_{1}^{\infty} \frac{\mathrm{d}x}{x} &= \lim_{u\to\infty}\int_{1}^{u}\frac{\mathrm{d}x}{x}\\ &= \lim_{u\to\infty}\Big[\ln x\Big]_1^u\\ &= \lim_{u\to\infty}(\ln u - \ln 1)\\ &= \infty,\end{aligned}$$

so that the integral diverges.

It also not hard to see that the integral in (4.1) diverges if we replace $\sin x$ by its absolute value.

However, the integral in (4.1) might converge, since the positive and negative bits might cancel to give a finite area. This is very similar to the fact that the harmonic series diverges but the alternating harmonic series converges.

Definition 4.2. *Let $f(x)$ be a continuous complex valued function on the real line,*

$$f\colon \mathbb{R} \longrightarrow \mathbb{C}.$$

Suppose that the improper integral

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

*converges. We say that this improper integral is **absolutely convergent** if*

$$\int_{-\infty}^{\infty} |f(x)|\,\mathrm{d}x$$

converges. Otherwise we say that the improper integral is ***conditionally convergent****.*

The integral in (4.1) is not absolutely convergent and we hope that it is conditionally convergent.

To show that the integral in (4.1) converges the trick is to pretend as though we know it does converge, compute the integral using contour integration and use the computation to justify convergence.

It is convenient to introduce:

Definition 4.3. *Let $f(x)$ be a continuous complex valued function on the real line*

$$f : \mathbb{R} \longrightarrow \mathbb{C}.$$

The ***Cauchy principal value*** *is the limit (assuming it exists):*

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x.$$

Note that this limit arises when we use the standard contour, we let R go infinity and we consider what happens along γ_1.

Observe two things. First if the improper integral

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

converges, then the Cauchy principal value exists and it is equal to the improper integral.

On the other hand, the improper integral might not exist and yet the Cauchy principal value does exist.

Example 4.4. *Consider the function $x \longrightarrow x$.*

First of all the improper integral

$$\int_{-\infty}^{\infty} x\,\mathrm{d}x$$

diverges. Indeed, by definition

$$\begin{aligned}\int_{-\infty}^{\infty} x\,\mathrm{d}x &= \lim_{u\to\infty, l\to-\infty} \int_{l}^{u} x\,\mathrm{d}x \\ &= \lim_{u\to\infty, l\to-\infty} \left[\frac{x^2}{2}\right]_{l}^{u} \\ &= \lim_{u\to\infty, l\to-\infty} \left(\frac{u^2}{2} - \frac{l^2}{2}\right).\end{aligned}$$

This limit diverges. For example, if we first let u go ∞ then the integral goes to ∞. If instead we first let l go ∞ then the integral goes to $-\infty$.

In fact if we coordinate how l and u approach ∞ then we can arrange for any limit we please. Compare this with a conditionally convergent series. If we are allowed to rearrange the terms then we can get any limit we please.

On the other hand, it is easy to see that the Cauchy principal value is zero:

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R} x\,\mathrm{d}x &= \lim_{R\to\infty}\left[\frac{x^2}{2}\right]_{-R}^{R} \\
&= \lim_{R\to\infty}\frac{R^2}{2}-\frac{R^2}{2} \\
&= \lim_{R\to\infty} 0 \\
&= 0.
\end{aligned}$$

Let us go back to computing

$$\int_{-\infty}^{\infty}\frac{x\sin(2x)}{x^2+3}\,\mathrm{d}x.$$

We integrate over the standard contour $\gamma=\gamma_1+\gamma_2$. As in Lecture 3, we change the integrand slightly

$$f(z)=\frac{ze^{2iz}}{z^2+3}.$$

$f(z)$ has isolated singularities at $\pm\sqrt{3}i$. Only $\sqrt{3}i$ belongs to the upper half plane. If $R>\sqrt{3}$ we capture this singularity. As this is a simple pole the residue is

$$\begin{aligned}
\operatorname{Res}_{\sqrt{3}i} f(z) &= \lim_{z\to\sqrt{3}i}\frac{(z-\sqrt{3}i)ze^{2iz}}{z^2+3} \\
&= \lim_{z\to\sqrt{3}i}\frac{ze^{2iz}}{z+\sqrt{3}i} \\
&= \frac{\sqrt{3}ie^{-2\sqrt{3}}}{2\sqrt{3}i} \\
&= \frac{e^{-2\sqrt{3}}}{2}.
\end{aligned}$$

The residue theorem implies that

$$\begin{aligned}
\int_\gamma \frac{ze^{2iz}}{z^2+3}\,\mathrm{d}z &= 2\pi i \operatorname{Res}_{\sqrt{3}i} f(z) \\
&= 2\pi i \frac{e^{-2\sqrt{3}}}{2} \\
&= \pi i e^{-2\sqrt{3}}.
\end{aligned}$$

We now attack the integral over γ_2, the semicircle of radius R in the upper half plane centred at 0. The problem is that it is not clear that this goes to zero as R goes to infinity. The length of γ_2 is πR and for the maximum value M of $|f(z)|$ we have

$$\begin{aligned}
|f(z)| &\leq \left|\frac{ze^{2iz}}{z^2+3}\right| \\
&= \frac{|ze^{2iz}|}{|z^2+3|} \\
&\leq \frac{R}{R^2-3}.
\end{aligned}$$

If we apply the usual estimate we just get

$$\begin{aligned}
\left|\int_{\gamma_2} \frac{ze^{2iz}}{z^2+3}\,\mathrm{d}z\right| &\leq LM \\
&\leq \frac{\pi R^2}{R^2-1},
\end{aligned}$$

which doesn't go to zero. It only goes to zero because of the e^{2iz} term and so using the LM-estimate just doesn't work.

Instead we use

Lemma 4.5 (Jordan's Lemma)**.**

$$\int_{\gamma_2} |e^{iz}||\mathrm{d}z| < \pi,$$

where γ_2 *is any semicircle in the upper half plane centred at the origin and* $|\mathrm{d}z| = \mathrm{d}s$ *represents the arclength parameter.*

We defer the proof of Jordan's Lemma (which isn't hard) and show how to use it for our purposes. We will need a slight improvement of the LM-estimate:

Lemma 4.6. *If γ is a piecewise differentiable curve and $h(z)$ is a continuous complex valued function on γ then*

$$\left|\int_\gamma h(z)\,\mathrm{d}z\right| \leq \int_\gamma |h(z)||\mathrm{d}z|.$$

Proof. This follows by realising the integral as a Riemann sum and applying the triangle inequality to the Riemann sum. □

We have

$$\begin{aligned}\left|\int_{\gamma_2} \frac{ze^{2iz}}{z^2+3}\,\mathrm{d}z\right| &\leq \int_{\gamma_2} \frac{|ze^{2iz}|}{|z^2+3|}\,|\mathrm{d}z| \\ &\leq \int_{\gamma_2} \frac{|Re^{2iz}|}{|R^2-3|}\,|\mathrm{d}z| \\ &= \frac{R}{R^2-3}\int_{\gamma_2} |e^{2iz}||\mathrm{d}z| \\ &< \frac{\pi R}{R^2-3},\end{aligned}$$

which goes to zero, as R goes to infinity.

We have

$$\int_{-R}^{R} \frac{xe^{2ix}}{x^2+3}\,\mathrm{d}x + \int_{\gamma_2} \frac{ze^{2iz}}{z^2+3}\,\mathrm{d}z = \pi i e^{-2\sqrt{3}}.$$

If we let R go to infinity then the RHS is constant and the second term on the LHS goes to zero. It follows that the limit as R goes to infinity of the first term exists, that is, the Cauchy principal value exists:

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{xe^{2ix}}{x^2+3}\,\mathrm{d}x = \pi i e^{-2\sqrt{3}}.$$

Taking the imaginary part of both sides we get

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{x\sin(2x)}{x^2+3}\,\mathrm{d}x = \pi e^{-2\sqrt{3}}.$$

But the integrand

$$\frac{x\sin(2x)}{x^2+3}$$

is even. In this case the existence of the Cauchy principal value shows that the improper integral converges.

Lemma 4.7. *Let $f(x)$ be a continuous complex valued function on the real line,*

$$f\colon \mathbb{R} \longrightarrow \mathbb{C}.$$

If $f(x)$ is even, that is, $f(x) = f(-x)$ then the Cauchy principal value

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x$$

exists if and only if the improper integral

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$$

converges, and in this case the two are the same:

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x.$$

Proof. If the improper integral exists then certainly the Cauchy principal value exists.

Now suppose that the Cauchy principal value exists. As $f(x)$ is even

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\int_{0}^{\infty} f(x)\,\mathrm{d}x.$$

On the other hand the Cauchy principal value is

$$\begin{aligned}\lim_{R\to\infty} \int_{-R}^{R} f(x)\,\mathrm{d}x &= \lim_{R\to\infty} \left(\int_{-R}^{0} f(x)\,\mathrm{d}x + \int_{0}^{R} f(x)\,\mathrm{d}x\right)\\ &= \lim_{R\to\infty} 2\int_{0}^{R} f(x)\,\mathrm{d}x\\ &= 2\int_{0}^{\infty} f(x)\,\mathrm{d}x.\end{aligned}$$ □

Putting all of this together shows that

$$\int_{-\infty}^{\infty} \frac{x\sin(2x)}{x^2+3}\,\mathrm{d}x = \pi e^{-2\sqrt{3}}.$$

We now turn to a proof of Jordan's Lemma:

Proof of (4.5). We just use the standard parametrisation

$$\gamma_2(\theta) = Re^{i\theta} \qquad \text{where} \qquad \theta \in [0, \pi].$$

In this case

$$|\mathrm{d}z| = R\mathrm{d}\theta.$$

On the other hand,

$$\begin{aligned}
|e^{iz}| &= |e^{iRe^{i\theta}}| \\
&= |e^{iR(\cos\theta + i\sin\theta)}| \\
&= |e^{iR\cos\theta - R\sin\theta}| \\
&= |e^{iR\cos\theta}||e^{-R\sin\theta}| \\
&= e^{-R\sin\theta}.
\end{aligned}$$

So we are reduced to showing that

$$\int_0^{\pi} e^{-R\sin\theta}\,\mathrm{d}\theta < \frac{\pi}{R}.$$

Consider the behaviour of $\sin\theta$ over the interval $[0, \pi]$. First of all we just need to worry about what happens over the interval $[0, \pi/2]$ by symmetry. Note that the graph of $\sin\theta$ over the interval $[0, \pi/2]$ never goes below the line connecting the endpoints, $(0, 0)$ and $(\pi/2, 1)$. It follows that

$$\sin\theta \geq \frac{2\theta}{\pi} \qquad \text{for any} \qquad \theta \in [0, \pi/2].$$

It follows that

$$\begin{aligned}
\int_0^{\pi} e^{-R\sin\theta}\,\mathrm{d}\theta &= 2\int_0^{\pi/2} e^{-R\sin\theta}\,\mathrm{d}\theta \\
&\leq 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,\mathrm{d}\theta \\
&= \frac{\pi}{R}\int_0^{R} e^{-t}\,\mathrm{d}t \\
&< \frac{\pi}{R}\int_0^{\infty} e^{-t}\,\mathrm{d}t \\
&= \frac{\pi}{R}.
\end{aligned}$$

□