3. Trigonometric over polynomial

Example 3.1. *Calculate*

$$\int_{-\infty}^{\infty} \frac{\cos(ax)\,\mathrm{d}x}{1+x^2} \qquad \textit{where} \qquad a>0.$$

We proceed in much the same way as before but we need to introduce a trick. The problem is that $\cos(az)$ isn't small on the semicircle of radius R. In fact $\cos(az)$ is the same as the function $\cosh(ay)$ on the imaginary axis and this is the sum of two exponential functions. If we use $\cos(az)$ we will get a top heavy fraction not a bottom heavy one.

The trick is to replace $\cos(az)$ by e^{iaz} and right at the end of the calculation we will recover the integral we are after.

We integrate over the standard contour and we use the notation established in lecture 2. The integrand is the function

$$f(z) = \frac{e^{iaz}}{1+z^2}.$$

This has isolated singularities at $\pm i$ and is otherwise holomorphic on the whole complex plane. If $R>1$ then i belongs to U but not $-i$.

We calculate the residue at i. As $f(z)$ has a simple pole at i we have

$$\begin{aligned}\operatorname{Res}_i f(z) &= \lim_{z\to i} \frac{(z-i)e^{iaz}}{1+z^2} \\ &= \lim_{z\to i} \frac{e^{iaz}}{z+i} \\ &= \frac{e^{-a}}{2i} \\ &= \frac{-e^{-a}i}{2}.\end{aligned}$$

The residue theorem then implies that

$$\begin{aligned}\int_\gamma \frac{e^{iaz}\mathrm{d}z}{1+z^2} &= 2\pi i \operatorname{Res}_i f(z) \\ &= 2\pi i \frac{-e^{-a}i}{2} \\ &= \pi e^{-a}.\end{aligned}$$

As usual,

$$\int_{\gamma_1} \frac{e^{iaz}\mathrm{d}z}{1+z^2} = \int_{-R}^{R} \frac{e^{iax}\mathrm{d}x}{1+x^2}.$$

Now $|e^{iax}| \leq 1$ and so the integral

$$\int_{-\infty}^{\infty} \frac{e^{iax}\mathrm{d}x}{1+x^2}.$$

is an improper integral which converges, since if x is large $1/x^2$ is bigger than the integrand. If we let R go to infinity we get the improper integral.

For the integral over γ_2 we try to bound the absolute value of the integral from above. The length L of γ_2 is πR. If $y = \mathrm{Im}(z) \geq 0$ then

$$\begin{aligned} |e^{iaz}| &= |e^{ia(x+iy)}| \\ &= |e^{iax}| \cdot |e^{-ay}| \\ &\leq 1. \end{aligned}$$

The maximum value M of the absolute value of $f(z)$ is

$$\begin{aligned} |f(z)| &= \left| \frac{e^{iaz}}{1+z^2} \right| \\ &= \frac{|e^{iaz}|}{|1+z^2|} \\ &\leq \frac{1}{R^2-1}. \end{aligned}$$

It follows that

$$\begin{aligned} \left| \int_{\gamma_2} \frac{e^{iaz}\mathrm{d}z}{1+z^2} \right| &\leq LM \\ &\leq \frac{\pi R}{R^2-1}, \end{aligned}$$

which goes to zero as R goes to zero.

It follows that

$$\int_{-\infty}^{\infty} \frac{e^{iax}\mathrm{d}x}{1+x^2} = \pi e^{-a}.$$

If we take the real part of both sides of this equation, we identify the integral we are after

$$\int_{-\infty}^{\infty} \frac{\cos(ax)\,\mathrm{d}x}{1+x^2} = \pi e^{-a}.$$

Note that if we take the imaginary part we get

$$\int_{-\infty}^{\infty} \frac{\sin(ax)\,\mathrm{d}x}{1+x^2} = 0.$$

There is another way to see this. The integrand

$$\frac{\sin(ax)}{1+x^2}$$

is an odd function. So

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin(ax)\,\mathrm{d}x}{1+x^2} &= \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(ax)\,\mathrm{d}x}{1+x^2} \\
&= \lim_{R\to\infty} \left(\int_{-R}^{0} \frac{\sin(ax)\,\mathrm{d}x}{1+x^2} + \int_{0}^{R} \frac{\sin(ax)\,\mathrm{d}x}{1+x^2} \right) \\
&= \lim_{R\to\infty} \left(-\int_{0}^{R} \frac{\sin(ax)\,\mathrm{d}x}{1+x^2} + \int_{0}^{R} \frac{\sin(ax)\,\mathrm{d}x}{1+x^2} \right) \\
&= \lim_{R\to\infty} 0 \\
&= 0.
\end{aligned}$$