26. Riemann Mapping Theorem: II

We end the course with an indication of how to prove the Riemann mapping theorem. In fact we sketch two approaches.

We begin with a basic fact that is used in both approaches:

Theorem 26.1. *Every harmonic function u on a simply connected region U has a harmonic conjugate v on U.*

The proof of (26.1) is not particularly hard. Recall that we constructed harmonic conjugates in Lecture 14 on rectangles and open disks by explicitly solving the Cauchy-Riemann equations. This involved integration over a carefully selected path. For a simply connected region U we start with a point $b \in U$ and we pick any path to a general point z. The fact that U is simply connected implies that the choice of path does not matter.

The first approach is relatively elementary, apart from one step. Let U be a simply connected region, not the whole complex plane. Pick a point $a \in U$. Let $\mathcal{F}$ be the set of all injective (or one to one) holomorphic functions from U to Δ that send a to 0 such that $f'(a) > 0$:

$$\mathcal{F} = \{\, f\colon U \longrightarrow \Delta \mid f \text{ is holomorphic, injective, } f(a) = 0 \text{ and } f'(a) > 0. \,\}$$

Note that we are looking for an element f of $\mathcal{F}$ that is also surjective (or onto). The basic idea is that if we are given an element g of $\mathcal{F}$ that is not surjective then we can improve g by increasing its image. Thus we are looking for one of the best elements of $\mathcal{F}$. The goal is to make this idea precise.

The first step is to show that $\mathcal{F}$ is non-empty, that is, to construct an element of $\mathcal{F}$.

Lemma 26.2. *Let U be a simply connected region and suppose that $c \notin U$.*

Then we can choose a holomorphic branch $h(z) = \sqrt{z-c}$ of the square root. $h(z)$ is injective, the map $h\colon U \longrightarrow h(U)$ is biholomorphic and $h(U)$ and $-h(U)$ are disjoint.

Proof. Consider the function

$$u(z) = \ln(z - c).$$

u is a harmonic function on U. (26.1) implies that u has a harmonic conjugate v. The function

$$g(z) = u(z) + iv(z) = \log(z - c)$$

defines a holomorphic branch of the logarithm. The function

$$h(z) = e^{g(z)/2}$$

is then a holomorphic branch of $\sqrt{z-c}$.

Suppose that $h(z_1) = h(z_2)$. Squaring both sides we get

$$\begin{aligned} z_1 - c &= h^2(z_1) \\ &= h^2(z_2) \\ &= z_2 - c. \end{aligned}$$

Adding a to both sides we get $z_1 = z_2$. Thus $h(z)$ is injective. It is clear that the derivative of h is nowhere zero. Thus $h\colon U \longrightarrow h(U)$ is biholomorphic.

Now suppose that

$$h(z_1) = -h(z_2)$$

is a common point of $h(U)$ and $-h(U)$. Squaring both sides we get

$$\begin{aligned} z_1 - c &= h^2(z_1) \\ &= h^2(z_2) \\ &= z_2 - c. \end{aligned}$$

Adding c to both sides we get $z_1 = z_2$. This is clearly nonsense and so $h(U)$ and $-h(U)$ are disjoint. □

Lemma 26.3. *$\mathcal{F}$ is non-empty.*

Proof. Suppose that the closed disk of radius ρ centred about d is contained in $h(U)$.

Then every point of $-h(U)$ is further than distance ρ from d:

$$|d + h(z)| > \rho \qquad \text{for all} \qquad z \in U.$$

It follows that

$$\frac{\rho}{|d + h(z)|} < 1 \qquad \text{for all} \qquad z \in U.$$

But then the holomorphic map

$$z \longrightarrow \frac{\rho}{h(z) + d}$$

sends U into Δ.

Suppose that $f(a) = b$. Consider the map

$$z \longrightarrow \frac{z - b}{1 - \bar{b}z}$$

This is a biholomorphic map of the unit disk that sends b to 0. The composition with the map above sends a to 0. Finally, if the argument of the derivative at a is θ then the rotation

$$z \longrightarrow e^{-i\theta} z$$

rotates the derivative back to a positive real. Thus $\mathcal{F}$ is non-empty. $\square$

Now we show that if $f \in \mathcal{F}$ and f is not surjective onto Δ then we can do better:

Lemma 26.4. *Let $V \subset \Delta$ be a simply connected region that contains 0.*

If $V \neq \Delta$ then there is a biholomorphic map

$$\psi \colon V \longrightarrow W$$

where $W \subset \Delta$, $\psi(0) = 0$ and $\psi'(0) > 1$.

Proof. Pick $b \notin V$. Let

$$g \colon \Delta \longrightarrow \Delta$$

be the biholomorphic map

$$g(z) = \frac{z - b}{1 - \bar{b}z}.$$

This sends b to 0 so that the image $g(V)$ is a simply connected region that does not contain 0. (26.2) implies that we can define a holomorphic branch of the square root function on $g(V)$, $h(z) = \sqrt{z}$. Finally let

$$f \colon \Delta \longrightarrow \Delta$$

be the biholomorphic map

$$f(z) = \frac{z - h(-b)}{1 - \overline{h(-b)}z}.$$

Then f sends $h(-b) = (h \circ g)(0)$ to 0. Thus the composition is a biholomorphic map of

$$\psi = f \circ h \circ g \colon V \longrightarrow W$$

such that $\psi(0) = 0$. Possibly applying a rotation we may assume that $\psi'(0) > 0$.

We have

$$g'(0) = 1 - |b|^2$$

and

$$f'(h(-b)) = \frac{1}{1 - |h(-b)|^2}.$$

Now

$$h^2(z) = z.$$

Thus

$$h'(z) = \frac{1}{2h(z)} \qquad \text{and so} \qquad h'(-b) = \frac{1}{2h(-b)}.$$

If $r = |-b|$ then $|h(-b)| = r^{1/2}$. It follows that

$$\begin{aligned}|\psi'(0)| &= |f'(h(-b))| \cdot |h'(-b)| \cdot |g'(0)| \\ &= \frac{1-|b|^2}{2|h(-b)| \cdot (1-|h(-b)|^2)} \\ &= \frac{1-r^2}{2r^{1/2} \cdot (1-r)} \\ &= \frac{1+r}{2r^{1/2}} \\ &= \frac{r^{-1/2}+r^{1/2}}{2} \\ &> 1.\end{aligned}$$

□

Note that the last inequality is an easy result from one variable calculus

$$x + \frac{1}{x} > 2 \qquad \text{for} \qquad x \in (0,1).$$

The best element of $\mathcal{F}$ is the function with the biggest derivative at 0. It is non-trivial result that $\mathcal{F}$ contains such an element:

Theorem 26.5. *There is an element $f \in \mathcal{F}$ such that if $g \in \mathcal{F}$ then*

$$f'(0) \geq g'(0).$$

We now give a proof of the Riemann mapping theorem:

Proof of 24.1. Pick $f \in \mathcal{F}$ with the largest derivative at 0. Suppose that f is not surjective. Pick $b \notin V = f(U)$. Then (26.4) implies we can find an injective holomorphic function

$$\psi \colon V \longrightarrow \Delta$$

such that $\psi'(0) > 1$. The composition $g = \psi \circ f$ is holomorphic, injective, $g(a) = 0$ and $g'(a) > 0$. Thus $g \in \mathcal{F}$. But

$$\begin{aligned}g'(a) &= \psi'(0) \cdot f'(a) \\ &> f'(a),\end{aligned}$$

which is not possible, by our choice of f. Thus f must be surjective so that f is biholomorphic. □

The second proof of the Riemann mapping theorem relies on the fact that we can solve Dirichlet's problem for U. As in the first proof, we may assume that U is bounded. There is no harm in assuming that $0 \in U$. Consider the continuous function $\ln|z|$ on the boundary. As we are assuming that we can solve Dirichlet's problem on U there is a harmonic function u on U extends to a continuous function on the boundary where it is $\ln |z|$. As U is simply connected it has a harmonic conjugate $v(z)$.

Consider

$$f(z) = ze^{-(u(z)+iv(z))}.$$

Then $f(z)$ is a holomorphic function and on the boundary we have

$$\begin{aligned} |f(z)| &= |ze^{-(u(z)+iv(z))}| \\ &= |z| \cdot |e^{-(u(z)}| \cdot |e^{-iv(z))}| \\ &= |z| \cdot |e^{-\log z}| \\ &= 1. \end{aligned}$$

Thus $|f(z)| < 1$ on U by the maximum principle. Thus $f\colon U \longrightarrow \Delta$ is holomorphic. $f(z)$ has only one zero, at zero.

Suppose that $b \in \Delta$. Pick $r > |b|$ and consider applying the argument principle to

$$f(z) - b$$

on the circle of radius r centred at 0 we see that $f(z)$ is bijective. Thus $f(z)$ is biholomorphic.