25. Schwarz-Christofell transformations

We consider a special but interesting case of the Riemann mapping theorem. We suppose that the boundary of the region U is a polygon, a finite number of straight line segments. We call such a region **polygonal**.

The polygon is described by its vertices $w_1, w_2, \dots, w_n$. The edges of the polygon are the line segments from w_1 to w_2, from w_2 to w_3, $\dots$, and from w_{n-1} to w_n. We suppose that with this orientation we are going around the polygon counterclockwise.

Somewhat counterintuitively instead of finding a biholomorphic map of the unit disk and the region U, we look for a biholomorphic map of the upper half plane $\mathbb{H}$ to U.

We are going to write down a differential equation satisfied by the solution. Most of the time it won't be possible to solve the differential equation using elementary functions. However, one can solve the differential equation numerically. If we start with an arbitrary region then one can approximate the region using a polygonal region.

It is clear that we want to map the real line, the boundary of the upper half plane, to the polygon. We first focus on how to arrange for the straight line segment between two points on the real axis (a, b) to map to a line segment connecting u to v. Notice that if f is a holomorphic map on the upper half plane that sends the interval (a, b) to the line segment connecting u to v then f extends to a holomorphic function across the interval (a, b) by the Schwarz reflection principle.

In particular it makes sense to write down the derivative of f. Consider the unit tangent vector to the real line at a point z of the interval (a, b). This gets sent to the unit tangent vector along the line segment connecting u to v. Recall that $f(z)$ turns the tangent vectors by the angle $\arg f'(z)$. The argument of the tangent vector to the real line is zero and the argument of the unit tangent vector to the line segment connecting u to v is fixed at some angle $\alpha \in [0, 2\pi)$. We have then

$$\alpha = \arg f'(z) \qquad \text{where} \qquad z \in (a, b).$$

In other words, the argument of the derivative of $f'(z)$ is constant.

Note that if we try

$$f'(z) = (z - a)^k$$

where k is a real number then the argument depends on whether $z < a$ or $z > a$. If $z < a$ then $z - a$ is a negative real number. The argument of $z - a$ is π and so the argument of $(z - a)^k$ is $k\pi$. This is a constant. If $z > a$ then $z - a$ is a positive real and the argument is zero. Raising to the kth power will not change the argument.

Note that $z = a$ is a point where the power function is not holomorphic. In fact we want to send a to a vertex of the polygon and so

As usual to define $(z-a)^k$ we have to cut the complex plane. We cut the complex plane with a vertical line starting at the point a.

Now if we multiply by a constant λ

$$f'(z) = \lambda(z-a)^k$$

this also changes the argument by a constant amount, the argument of λ.

Since we want a transition at a and b the next thing to try is

$$f'(z) = \lambda(z-a)^{-k}(z-b)^{-l}.$$

Now there are three regions along the real axis. $z < a$, $a < z < b$ and $z > b$. We put two cuts, two vertical lines, one at a and one at b. When $z < a$ we get an argument

$$\arg f'(z) = \arg \lambda - k\pi - l\pi.$$

When $a < z < b$ the argument is

$$\arg f'(z) = \arg \lambda - l\pi.$$

Finally when $z > b$ the argument is

$$\arg f'(z) = \arg \lambda.$$

The easiest way to keep track of what is going on is to consider the change in argument as we cross a branch point. At the start when z is large and negative we describe a straight line with a slope given by the angle $\arg \lambda - k\pi - l\pi$. Then we turn through an angle of $k\pi$ at the vertex u and we describe the straight line with angle $\arg \lambda - l\pi$. Finally when we pass through b the angle increases by $l\pi$ and so we turn through an angle of $l\pi$ at the vertex v and from there we describe a sraight line of angle $\arg \lambda$.

Note that we use negative exponents so that as we pass through a branch point the argument increases.

Suppose that the vertices of the region correspond to the real numbers $a_1, a_2, \dots, a_{n-1}$. We suppose that a_i is mapped to w_i and ∞ is mapped to w_n. In this case the real line is the union of the intervals $(-\infty, a_1)$, (a_1, a_2), $\dots$, (a_{n-2}, a_{n-1}) and (a_{n-1}, ∞) and the points $a_1, a_2, \dots, a_{n-1}$. The interval $[a_i, a_{i+1}]$ is supposed to be mapped to the line segment connecting w_i to w_{i+1}. Again we orient the numbers so that $a_1 < a_2 < a_3 < \cdots < a_{n-1}$.

Consider the vertices of the polygon $w_1, w_2, \dots, w_n$. The line segment from w_n to w_1 makes an angle with the real line. If we traverse this line segment then when we get to w_1 and start to traverse the line segment from w_1 to w_2 then we have to turn through an angle $k_1\pi$. Then when we get to the vertex w_2 we have to turn through an angle $k_2\pi$, and so on.

This gives a differential equation that the biholomorphic function f mapping the upper half plane to U must satisfy:

$$f'(z) = \lambda(z - a_1)^{-k_1}(z - a_2)^{-k_2} \dots (z - a_{n-1})^{-k_{n-1}}$$

To make sense of the RHS we must put cuts, vertical lines in the lower half plane starting at the points $a_1, a_2, \dots, a_{n-1}$. Let V be the region you get by deleting these cuts, so that the RHS is a holomorphic function on the region V.

Note that as go all the way around the polygon we have to turn through a full 2π. The real numbers $k_i \in (-1, 1)$ and we have

$$\begin{aligned} 2\pi &= k_1\pi + k_2\pi + \dots + k_n\pi \\ &= (k_1 + k_2 + k_3 + \dots + k_n)\pi. \end{aligned}$$

Thus

$$k_1 + k_2 + \dots + k_n = 2.$$

Note that the last term, k_n, corresponds to what happens at ∞.

To solve the differential equation, just pick a point $b \in V$. Given another point $z \in V$, define an integral

$$F(z) = \int_b^z (s - a_1)^{-k_1}(s - a_2)^{-k_2} \dots (s - a_{n-1})^{-k_{n-1}} \, \mathrm{d}s$$

by integrating along a path connecting b to z. As the region V is simply connected it follows that the integral does not depend on the choice of path. $F(z)$ is a holomorphic function on V.

As icing on the cake, we check that $F(z)$ is continuous at the points $a_1, a_2, \dots, a_{n-1}$. The only part of $f'(z)$ which is not holomorphic at a_i is the term $(z - a_i)^{-k_i}$. The other terms are holomorphic. Let the product of these terms be $\phi(z)$. We may write.

$$\phi(z) = \phi(a_i) + (z - a_i)\psi(z),$$

where $\psi(z)$ is holomorphic in a neighbourhood of a_i. It follows that

$$f'(z) = (z - a_i)^{-k_i} + (z - a_i)^{1-k_i}\psi(z).$$

Note that $1 - k_i > 0$ so that the last term is continuous at a_i. Therefore the integral is continuous. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}z}(z - a_i)^{1-k_i} = (1 - k_i)(z - a_i)^{-k_i}.$$

This implies the integral is continuous at a_i.

The general form of the solution to the differential equation is then

$$f(z) = \lambda F(z) + \mu$$

where $\mu \in \mathbb{C}$ is a constant.

As already mentioned there is no simple general formula for the solution $f(z)$. It is an interesting exercise to count constants. We may assume that $b = 0$, $\lambda = 1$ and $\mu = 0$. It is enough to show we get a polygon P' which is similar to P. In this case adjusting μ matches up one of the vertices and adjusting λ both rotates and rescales.

Now the choice of k_i determines the angles of the polygon. Once we have fixed these the polygon P' has the same angles as P. It remains to determine the real numbers $a_1, a_2, \dots, a_{n-1}$. Now just because P and P' have the same angles does not mean they are similar (this is only true for triangles). In fact we need to fix $n-2$ corresponding sides of P and P' and require that the ratios between these common sides are all equal. This gives $n-3$ equations for $a_1, a_2, \dots, a_{n-1}$.

Thus we are free to specify two of these $n-1$ real numbers and the remaining $n-3$ numbers are determined.