24. Riemann mapping theorem

Theorem 24.1. *If $U \subset \mathbb{C}$ is a simply connected region then either $U = \mathbb{C}$ or U is biholomorphic to the unit disk Δ.*

Recall that U is simply connected if every closed path can be continuously shrunk to a point. Intuitively this says that U has no holes. Every open disk is simply connected but an annulus is never simply connected. It is theorem that a region is simply connected if and only if the complement in the Riemann sphere is connected.

If we think about trying to construct a biholomorphic map $f\colon U \longrightarrow \Delta$ the first thing to realise is that f is not unique. If we compose f with a biholomorphic map α of Δ to itself we get another biholomorphic map $g = \alpha \circ f$.

Recall that α has the form

$$z \longrightarrow e^{i\theta} \frac{z-a}{1-\bar{a}z}.$$

α is determined by two pieces of information, a, which is the inverse image of 0 and the argument θ of the derivative at zero, $e^{i\theta}$.

To make f unique, we pick a point a in U and we require that $f(a) = 0$. Then we require the derivative $f'(a) > 0$ is real and positive. In terms of α, this fixes a and $e^{i\theta}$.

Corollary 24.2. *If $U \subset \mathbb{P}^1$ is a simply connected region in the extended complex plane then either*

(1) $U = \mathbb{P}^1$, or
(2) U is biholomorphic to $\mathbb{C}$, or
(3) U is biholomorphic to the unit disk Δ.

Proof. Let Z be the complement of U. Then Z is empty if and only if $U = \mathbb{P}^1$.

Suppose that Z is not empty. If $a \in Z$ then pick a Möbius transformation that sends a to ∞. For example,

$$z \longrightarrow \frac{1}{z-a}$$

will do. The image of U under this Möbius transformation is biholomorphic to U. Thus we may assume that $\infty \in Z$. But then $U \subset \mathbb{C}$ and the Riemann mapping theorem implies that (2) or (3) holds. □