23. Through the looking glass

It is a somewhat surprising observation that we can extend the reflection principle to a much broader class of reflections.

Definition 23.1. *Let γ be a curve.*

*We say that γ is **analytic** if for every point q of γ we can find an open disk V centred around q and a biholomorphic map with an open disk U centred about a point $p \in \mathbb{R}$ such that γ matches up with the real axis in U.*

Example 23.2. *If γ is the arc of a circle then γ is analytic.*

Indeed, let us first suppose that γ is the arc of the unit circle, the boundary of the unit disk. Suppose a is a point of γ. First suppose that $a \neq -1$. The Möbius transformation

$$z \longrightarrow i\frac{1-z}{1+z}$$

sends the unit circle minus -1 to the real line.

If $a = -1$ then we can move a to another point of the unit circle and use the Möbius transformation above. Now suppose γ is the arc of a circle of radius R centred at a. The Möbius transformation

$$z \longrightarrow \frac{z-a}{R}$$

sends this circle to the unit circle and composing with the first Möbius transformation we see that γ is analytic.

Let us suppose we use ζ as the coordinate on U and z is the coordinate on V. By assumption there is a correspondence between V and U and so we may think of z as a function of ζ,

$$z = z(\zeta) = \alpha(\zeta).$$

Complex conjugation on U defines a symmetry of V,

$$z \longrightarrow z^* = z(\bar{\gamma}) = \alpha(\bar{\gamma})$$

This symmetry is an **involution** in the sense that if you apply the symmetry twice nothing happens:

$$z^{**} = z.$$

This follows as complex conjugation is an involution. The symmetry $z \longrightarrow z^*$ fixes the analytic arc γ and nothing else, since complex conjugation fixes the real axis and nothing else. $V \setminus \gamma$ has two connected components, as $U \setminus U^0 = U^+ \cup U^-$ has two connected components, the part above the real axis and the part below. The symmetry $z \longrightarrow z^*$ switches these two components.

We call the two components of V the **sides** of γ and we call the symmetry $z \longrightarrow z^*$ **reflection across** γ.

Lemma 23.3. *Reflection across γ is unique.*

Proof. Suppose that there were two antiholomorphic maps which fixed γ, α and β, where α is reflection across γ. Then the composition $\alpha \circ \beta$ is conformal, as it is the composition of two antiholomorphic maps. Thus the composition is holomorphic and fixes γ. But then the composition is the identity. As α is its own inverse, it follows that $\beta = \alpha$. □

Using (23.3) It follows that we may define reflection across any analytic curve in a region U. We do this locally about any point of γ and then we use (23.3) to patch the local constructions.

Example 23.4. *Suppose that γ is the unit circle.*

In this case

$$\mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \zeta \longrightarrow \frac{\zeta - i}{\zeta + i}$$

sends the upper half plane to the unit disk and the real line to the unit circle.

Reflection across the unit circle corresponds to complex conjugation:

$$\begin{aligned} z^* &= \frac{\bar{\zeta} - i}{\bar{\zeta} + i} \\ &= \frac{\overline{\zeta + i}}{\overline{\zeta - i}} \\ &= \overline{\frac{\zeta + i}{\zeta - i}} \\ &= \overline{\frac{1}{z}} \\ &= \frac{1}{\bar{z}}. \end{aligned}$$

We already saw this map in the construction of the Poisson kernel. If

$$z = re^{i\theta} \qquad \text{then} \qquad z^* = \frac{1}{r}e^{i\theta}.$$

The reflection z^* of z has the same argument as z but the modulus is the reciprocal of the original modulus. Clearly the two sides of the reflection are the inside and the outside of the circle, the unit disk and the complement of the closed unit disk. Although note that the centre of the circle is sent to the point at infinity, not a point in the finite plane.

It turns out that we can reflect a holomorphic function $f(z)$ about an analytic arc γ if the values of $f(z)$ approach another analytic arc σ as z approaches γ. In this case the extension satisfies

$$f(z^*) = f(z)^*.$$

In this formula, note that the * on the LHS represents reflection across γ and the * on the RHS represents reflection across σ. We state a version when both γ and σ are the unit circle:

Theorem 23.5. *Let $f(z)$ be a holomorphic function on the unit disk*

$$f\colon \Delta \longrightarrow \mathbb{C}.$$

If $|f(z)|$ approaches 1 as $|z|$ approaches 1,

$$\lim_{|z|\to 1} |f(z)| = 1,$$

then $f(z)$ extends to a meromorphic function on the whole complex plane with the property that

$$f\left(\frac{1}{\bar{z}}\right) = \frac{1}{\overline{f(z)}}.$$

Proof. Pick a point a of the unit circle. Possibly rotating the unit circle, there is no harm in assuming $a \neq -i$. We may also assume that $f(z)$ does not approach $-i$ as z approaches a. Let

$$\alpha\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \alpha(\zeta) = \frac{\zeta - i}{\zeta + i}.$$

Then α is a Möbius transformation that sends the upper half plane $\mathbb{H}$ to the unit disk Δ. The inverse transformation is the Möbius transformation

$$\beta\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad \beta(z) = i\frac{z+1}{1-z}.$$

Consider the function

$$g = \beta \circ f \circ \alpha \colon \mathbb{H} \longrightarrow \mathbb{C}$$

Then g is a holomorphic map, as it is a composition of holomorphic maps. As we approach the real axis $g(z)$ approaches a real number, that is, the imaginary part goes to zero. The reflection principle applied to g implies that g extends to a holomorphic map on the whole complex plane and

$$g(\bar{\zeta}) = \overline{g(\zeta)}.$$

It follows that $f = \alpha \circ g \circ \beta$ extends to the whole complex plane as a meromorphic function. Since complex conjugation for ζ corresponds to inversion in the unit circle for z, it follows that

$$f\left(\frac{1}{\bar{z}}\right) = \frac{1}{\overline{f(z)}}. \qquad \square$$

Of course there are many variations on a theme for (23.5). We might start with a region $U = U^*$ symmetric about the unit circle and a holomorphic function on the side inside the unit disk whose modulus approaches 1 as we approach the boundary. Reflection in the circle extends this function to a holomorphic function on the whole of U which satisfies the equation above.

Or we might have two circles of different radii. In this case we can extend a holomorphic function on the open disk corresponding to the first circle to a holomorphic function on the whole complex plane but now the functional equation (that is, the equation the function satisfies) changes.