22. The Schwarz Reflection Principle

First a little bit of notation.

Definition 22.1. *The **reflection** of a region U about the real axis is*

$$U^* = \{\, \bar{z} \mid z \in U \,\}.$$

If $u\colon U \longrightarrow \mathbb{R}$ is a real valued function on U then define

$$u^*\colon U^* \longrightarrow \mathbb{R} \qquad \text{by} \qquad u^*(z) = u(\bar{z}).$$

*We say that U is **symmetric about the real axis** if $U = U^*$.*

Example 22.2. *The unit circle is symmetric about the real axis.*

Lemma 22.3. *If u is a harmonic function on a region U then u^* is harmonic on U^*.*

Proof. There are two ways to see this.

For the first way note that u satisfies the mean value property as u is harmonic. But then u^* also satisfies the mean value property which implies that u^* is harmonic.

For the second note that complex conjugation replaces (x, y) by $(x, -y)$. This leaves the Laplacian unchanged; indeed nothing happens to x and so

$$\frac{\partial^2}{\partial x^2}$$

is unchanged. The change in sign in y flips the sign of

$$\frac{\partial}{\partial y}$$

and so it also leaves

$$\frac{\partial^2}{\partial y^2}$$

unchanged. □

Theorem 22.4. *Let U be a region symmetric about the real axis and let*

$$U^+ = \{\, z \in U \mid \operatorname{Im} z > 0 \,\}$$

be the part of U in the upper half plane $\mathbb{H}$.

If $u(z)$ is a harmonic function on U^+ such that

$$\lim_{\operatorname{Im}(z)\to 0^+} u(z) = 0$$

as z approaches the real axis from above then $u(z)$ extends to a harmonic function $u = u^e$ on U which satisfies

$$u(\bar{z}) = -u(z) \qquad \text{for all} \qquad z \in U.$$

Proof. We can decompose U into three disjoint sets

$$U^+ = \{ z \in U \mid \operatorname{Im} z > 0 \} \quad U^- = \{ z \in U \mid \operatorname{Im} z < 0 \} \quad \text{and} \qquad U^0 = \{ z \in U \mid \operatorname{Im} z = 0 \}.$$

We extend u to the whole of U in the obvious way:

$$u^e(z) = \begin{cases} u(z) & \text{if } z \in U^+ \\ -u(\bar{z}) & \text{if } z \in U^- \\ 0 & \text{if } z \in U^0. \end{cases}$$

Then the restriction of u^e to U^- is $-u^*$. It follows that u^e is harmonic on U^-.

It is then clear that u^e is continuous on U. We now check that u^e is harmonic. It suffices to check that u^e has the mean value property. Let $z_0 \in U$. If $z_0 \in U^+ \cup U^-$ then it is clear that u satisfies the mean value property for small enough disks centred around z_0.

Now suppose that $z_0 \in U^0$. If we have a disk centred around z_0 then half of the disk is above the real axis and half of it is below. Since the integral around the bottom half is minus the integral around the top half, the average value is zero. But this is also the value of $u^e(z_0)$. It follows that u^e satisfies the mean value property. In particular u^e is harmonic. □

There is also a reflection principle for holomorphic functions.

Lemma 22.5. *If f is a holomorphic function on a region U then*

$$g(z) = \overline{f(\bar{z})}$$

is holomorphic on U^*.

Proof. There are three ways to see this.

For the first we may write $f(z) = u(z) + iv(z)$. In this case $g(z) = u(\bar{z}) - iv(\bar{z}) = p(z) + iq(z)$. We have

$$\begin{aligned} p_x &= u_x \\ &= v_y \\ &= q_y. \end{aligned}$$

Here we used the Cauchy-Riemann equations for u and v to get from the first line to the second line. Note that there are two minuses to get from the second line to the third line.

We also have

$$\begin{aligned} p_y &= -u_y \\ &= v_x \\ &= -q_x. \end{aligned}$$

Thus p and q satisfy the Cauchy-Riemann equations and $g(z)$ is holomorphic.

For the second we may write down a power series expansion for $f(z)$ locally about a point a, using the fact that holomorphic implies analytic:

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \dots.$$

In this case

$$g(z) = b_0 + b_1(z-b) + b_2(z-b)^2 + b_3(z-b)^3 + \dots,$$

where $b = \bar{a}$ and $b_i = \bar{a}_i$, with the same radius of convergence. Thus $g(z)$ is analytic, so that it is holomorphic.

For the third, assume that $f(z)$ has nowhere vanishing derivative. Then $f(z)$ is conformal. Now complex conjugation reverses angles, so that $g(z)$ is conformal, as we reverse angles twice, once for $\bar{z}$ and once for $\bar{f}$. Thus $g(z)$ is holomorphic, except at the points corresponding to the zeroes of $f'(z)$. Now the zeroes of $f(z)$ are isolated, so that $g(z)$ has isolated singularities. As $g(z)$ is continuous near the singular points, it is certainly bounded, and so $g(z)$ has removable singularities. Thus $g(z)$ is holomorphic. □

Theorem 22.6. *Let U be a region symmetric about the real axis and let*

$$U^+ = \{\, z \in U \mid \operatorname{Im} z > 0 \,\}$$

be the part of U in the upper half plane $\mathbb{H}$.

If $f(z)$ is a holomorphic function on U^+ such that

$$\lim_{\operatorname{Im}(z)\to 0^+} \operatorname{Im} f(z) = 0$$

as z approaches the real axis from above then $f(z)$ extends to a holomorphic function $f = f^e$ on U which satisfies

$$f(\bar{z}) = \overline{f(z)} \qquad \textit{for all} \qquad z \in U.$$

Proof. First extend $f(z)$ to a holomorphic function on U^- by using (22.5):

$$f^e(z) = \overline{f(\bar{z})} \qquad \text{for all} \qquad z \in U^-.$$

We have to check that we can extend f across the real line.

Suppose that we write $f(z) = u(z) + iv(z)$ on U^+. By assumption

$$\lim_{\operatorname{Im}(z)\to 0^+} v(z) = 0$$

and so $v(z)$ extends to a harmonic function on the whole of U such that

$$v(\bar{z}) = -v(z).$$

Note that v is the imaginary part of f on U^-.

Fix a point $a \in U^0$ and pick a small disk D centred around a. As $v(z)$ is harmonic it has a harmonic conjugate on D. This harmonic conjugate differs by a constant from $u(z)$ on $D \cap U^+$. Thus $u(z)$ extends to a harmonic function on D and so $f(z)$ extends to a holomorphic function on D.

By the same token, the function $f^e(z) = \overline{f(\bar{z})}$ is also holomorphic on D, it has the same inaginary part as $f(z)$ and agrees with $f(z)$ on $D \cap U^0$. Thus

$$f(\bar{z}) = \overline{f(z)} \qquad \text{for all} \qquad z \in D.$$

This implies that f^e is holomorphic on the whole of U. $\square$