21. The mean value property

We recall a couple of basic properties of continuous functions enjoyed by harmonic functions, see lecture 14.

We say that a continuous function h on a region U has the *mean value property* if its value at the centre of a circle is the average value on the boundary,

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})\,\mathrm{d}\theta \qquad \text{where} \qquad r \in (0, \rho).$$

for any open disk centred at a of radius ρ contained in U.

We also saw a proof of the fact that any function which has the mean value property on U satisfies the maximum principle on U.

Theorem 21.1. *If $h(z)$ is a continuous function on a region U then $h(z)$ is harmonic if and only if satisfies the mean value property.*

Proof. We already saw that any harmonic function satisfies the mean value property.

Suppose that $h(z)$ satisfies the mean value property. Let V be an open disk whose boundary is contained in U. It is enough to show that $h(z)$ is harmonic on V. Let $F(z)$ be the restriction of $h(z)$ to the boundary of V.

Then $F(z)$ is a continuous function on a circle. As we can solve Dirichlet's problem for the circle, using the Poisson kernel, we can find a harmonic function u on V which has a continuous extension to $V \cup \partial V$ and which equals $F(z)$ on the boundary.

As u is harmonic it satisfies the mean value property. Thus the difference $v = u - h$ satisfies the mean value property. In particular $u - h$ satisfies the maximum principle. By assumption F agrees with h on the boundary and u agrees with F on the boundary. Thus u and h agree on the boundary so that $v = 0$ on the boundary.

Thus $v \leq 0$ on V by the maximum principle. By the minimum principle, the maximum principle applied to $h - u$ which also satisfies the mean value property, $v \geq 0$ on V. Thus $v = 0$ on V. But then $h = u$ and so h is harmonic on V as u is harmonic on V. □