20. The Poisson kernel and Dirichlet problem

Note that we haven't solved Dirichlet's problem yet. We started with a holomorphic function on the closed unit disk. In other words, we started by assuming we had a harmonic function on the closed unit disk and we derived a formula for it using the Poisson kernel.

Now suppose we start with a continuous function h on the unit circle and we try to solve the Dirichlet problem for h. It is a somewhat amazing fact that we can even deal with a slightly larger class of functions. We can allow h to have finitely many discontinuities. In this case we cannot expect $u(r, \theta)$ to extend to the boundary as a continuous function but we can expect it to extend away from the finitely many points where h is not continuous.

Definition 20.1. *Let h be a piecewise continuous function on the unit circle and let ζ be a point of the unit circle. The **average value** of h at $\zeta = e^{i\theta_0}$ is*

$$\frac{1}{2}\left(\lim_{\theta \to \theta_0^-} h(e^{i\theta}) + \lim_{\theta \to \theta_0^+} h(e^{i\theta})\right).$$

Theorem 20.2. *Let h be a piecewise continuous real valued function on the unit circle.*

Define a function $u(r, \theta)$ on the unit disk by the formula

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)h(e^{i\phi})}{1 - 2r\cos(\phi - \theta) + r^2}\,\mathrm{d}\phi.$$

Then u is a harmonic function on the unit disk, it extends to a continuous function on the closed unit disk minus the points where h is discontinuous and it is equal to h on the unit circle, minus the points where h is discontinuous. If $\zeta = e^{i\theta}$ is a point on the unit disk where h is not continuous then

$$\lim_{r \to 1} u(r, \theta_0)$$

is the average value of h at $\zeta = e^{i\theta_0}$.

Example 20.3. *Suppose that*

$$h(e^{i\phi}) = \begin{cases} 0 & \text{if } \phi \in (0, \pi) \\ 1 & \text{if } \phi \in (\pi, 2\pi). \end{cases}$$

Let us find a harmonic function $u(r, \theta)$ on this unit disk with these boundary values. There are two ways to solve this problem. The first uses the Poisson integral.

We have

$$\begin{aligned} u(r,\theta) &= \int_0^{2\pi} P_r(\phi-\theta)h(e^{i\phi})\,\mathrm{d}\phi \\ &= \int_\pi^{2\pi} P_r(\phi-\theta)\,\mathrm{d}\phi. \end{aligned}$$

We want to integrate

$$P_r(\psi) = \frac{1-r^2}{1-2r\cos(\psi)+r^2}.$$

One can check that the derivative of

$$2\arctan\left(\frac{1+r}{1-r}\tan\frac{\psi}{2}\right)$$

with respect to ψ is $P_r(\psi)$. It follows that

$$\pi u(r,\theta) = \arctan\left(\frac{1+r}{1-r}\tan\frac{2\pi-\theta}{2}\right) - \arctan\left(\frac{1+r}{1-r}\tan\frac{\pi-\theta}{2}\right).$$

If take tan of both sides and use some trigonometric identities the RHS reduces to

$$u(r,\theta) = \frac{1}{\pi}\arctan\left(\frac{1-r^2}{2r\sin\theta}\right) \qquad \text{where} \qquad \arctan t \in [0,\pi].$$

The last expression speficies the choice of arctan we are using. It is clear that $u(r,\theta)\in[0,1]$.

It is interesting to check the boundary values. If $\theta \neq 0$ and $\theta \neq \pi$ then the expression

$$\frac{1-r^2}{2r\sin\theta}$$

approaches zero. If $\theta \in (0,\pi)$ it approaches from above and the angle given by the arctangent approaches 0. If $\theta\in(\pi,2\pi)$ it approaches from below and the angle given by the arctangent approaches π.

If $\theta = 0$ or $\theta=\pi$ then the angle given by the arctangent is $\pi/2$, which gives the value $1/2$ for the limit of u. $1/2$ is of course the average value at these two angles.

Note that the value of u at the origin is the average value of h on the circle. In our case the average value is $1/2$ and this is indeed the value of u at the origin.

For the second approach we need a simple:

Lemma 20.4. *Let $\alpha\colon U \longrightarrow V$ be a holomorphic function between two regions.*

If $v\colon V\longrightarrow \mathbb{R}$ is a harmonic function on V then $v\circ\alpha\colon U\longrightarrow\mathbb{R}$ is a harmonic function on U.

Proof. As this problem is local there is no harm in assuming that v is the real part of the holomorphic function $g\colon V \longrightarrow \mathbb{C}$. Then

$$f = g \circ \alpha\colon U \longrightarrow \mathbb{C}$$

is a holomorphic function and u is the real part of f. $\square$

Let

$$\alpha\colon \Delta \longrightarrow \mathbb{H}$$

be the biholomorphic map

$$\alpha(z) = i\frac{1-z}{1+z}.$$

This maps the unit circle minus the point -1 to the real axis. The upper half semicircle gets sent to $(0,\infty)$ and the lower half semicircle gets sent to $(-\infty,0)$. The original problem is transformed to the problem of finding a harmonic function on the upper half plane that extends to a continuous function on the closure minus zero and that is 0 on $(0,\infty)$ and 1 on $(-\infty,0)$.

Consider the principal value of the logarithm

$$\frac{1}{\pi}\operatorname{Log} w = \frac{1}{\pi}\ln\rho + i\frac{1}{\pi}\phi,$$

where $w = \rho e^{i\phi}$, $\rho > 0$ and $\phi \in (0,\pi)$. This is holomorphic on the upper half plane. Note that the imaginary part is a harmonic function with the correct boundary values.

The imaginary part is

$$\frac{1}{\pi}\arctan\left(\frac{v}{u}\right),$$

where $w = u + iv$.

Now

$$\begin{aligned} w &= i\frac{1-z}{1+z} \\ &= i\frac{(1-z)(1+\bar{z})}{|1+z|^2} \\ &= i\frac{1-|z|^2+\bar{z}-z}{|1+z|^2} \\ &= i\frac{1-x^2-y^2-2iy}{(x+1)^2+y^2} \\ &= \frac{2y+i(1-x^2-y^2)}{(x+1)^2+y^2} \\ &= u+iv. \end{aligned}$$

It follows that

$$u = \frac{2y}{(x+1)^2 + y^2} \qquad \text{and} \qquad v = \frac{1 - x^2 - y^2}{(x+1)^2 + y^2}.$$

Thus

$$\frac{1}{\pi} \arctan\left(\frac{1 - x^2 - y^2}{2y}\right).$$

This is clearly compatible with the previous solution expressed in polar coordinates.

It is possible to interpret (20.3) as various instances of problems from physics. One can either imagine a thin sheet of metal in the shape of a circle. The upper side is cooled to constant temperature 0 and the lower side is heated to constant temperature 1. The solution to Laplace's equation then represents the distribution of heat in the metal in steady state. Or one can imagine a long thin cylinder in space. The electric potential on the upper half of the cylinder is zero and the potential on the lower side of the cylinder is one. The two plates are separated by a thin layer of insulation. As the cylinder is long and thin one can just solve Laplace's equation for a two dimensional slice with cross-section the unit circle.

We now turn to a proof of (20.2). The fact that $u(r, \theta)$ is harmonic is quite straightforward. Indeed $u(r, \theta)$ is harmonic if it satisfies the polar form of Laplace's equation, which involves partial derivatives with respect to r and θ:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Consider

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) h(e^{i\phi})}{1 - 2r\cos(\phi - \theta) + r^2} \, \mathrm{d}\phi.$$

As we don't integrate over either r or θ, if we differentiate with respect to r and θ then we can bring the partial derivatives inside the integral sign. As $h(e^{i\phi})$ does not depend on r and θ, the Laplacian applied to

$$h(e^{i\phi}) P_r(\phi - \theta)$$

is the same as the Laplacian applied to $P_r(\phi - \theta)$, multiplied by $h(e^{i\phi})$. But if we fix ϕ then $P_r(\phi - \theta)$ is a harmonic function of r and θ, by (19.2.c). So the Laplacian applied to the integrand is zero and so the Laplacian applied to the integral is zero. It follows that $u(r, \theta)$ is harmonic.

We now argue that $u(r, \theta)$ behaves properly on the boundary. Pick a point $\zeta = e^{i\phi_0}$ on the boundary where h is continuous. We have to show that if z is sufficiently close to ζ then u is sufficiently close to h.

This means given $\epsilon > 0$ we have to find $\delta > 0$ such that if $|z - \zeta| < \delta$ then $|u(r, \theta) - h(e^{i\phi})| < \epsilon$. Note that

$$\begin{aligned}
u(r, \theta) - h(e^{i\phi_0}) &= \frac{1}{2\pi} \int_0^{2\pi} P_r(\phi - \theta) h(e^{i\phi}) \, \mathrm{d}\phi - h(e^{i\phi_0}) \frac{1}{2\pi} \int_0^{2\pi} P_r(\phi - \theta) \, \mathrm{d}\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} P_r(\phi - \theta) h(e^{i\phi}) \, \mathrm{d}\phi - \frac{1}{2\pi} \int_0^{2\pi} P_r(\phi - \theta) h(e^{i\phi_0}) \, \mathrm{d}\phi \\
&= \frac{1}{2\pi} \int_0^{2\pi} P_r(\phi - \theta)(h(e^{i\phi}) - h(e^{i\phi_0})) \, \mathrm{d}\phi \\
&= I_1 + I_2.
\end{aligned}$$

We are going to break up the integral into two pieces. The first integral is over a small arc centred around ϕ_0 and the second integral is over the rest of the unit circle. The first integral will be small as $h(e^{i\phi}) - h(e^{i\phi_0})$ is small. The second integral will be small as the integral of the Poisson kernel is small.

We may pick $\alpha > 0$ so that if

$$|h(e^{i\phi}) - h(e^{i\phi_0})| < \frac{\epsilon}{2} \qquad \text{whenever} \qquad |\phi - \phi_0| < \alpha.$$

Let

$$\begin{aligned}
I_1 &= \frac{1}{2\pi} \int_{\phi_0 - \alpha}^{\phi_0 + \alpha} P_r(\phi - \theta)(h(e^{i\phi}) - h(e^{i\phi_0})) \, \mathrm{d}\phi \\
I_2 &= \frac{1}{2\pi} \int_{\phi_0 + \alpha}^{\phi_0 - \alpha + 2\pi} P_r(\phi - \theta)(h(e^{i\phi}) - h(e^{i\phi_0})) \, \mathrm{d}\phi.
\end{aligned}$$

We have

$$\begin{aligned}
|I_1| &\leq \frac{1}{2\pi} \int_{\phi_0 - \alpha}^{\phi_0 + \alpha} P_r(\phi - \theta) |h(e^{i\phi}) - h(e^{i\phi_0})| \, \mathrm{d}\phi \\
&\leq \frac{\epsilon}{2} \left(\frac{1}{2\pi} \int_{\phi_0 - \alpha}^{\phi_0 + \alpha} P_r(\phi - \theta) \, \mathrm{d}\phi \right) \\
&\leq \frac{\epsilon}{2} \left(\frac{1}{2\pi} \int_0^{2\pi} P_r(\phi - \theta) \, \mathrm{d}\phi \right) \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

For I_2, note that $|h(e^{i\phi}) - h(e^{i\phi_0})|$ is bounded on the circle:

$$|h(e^{i\phi}) - h(e^{i\phi_0})| \leq M.$$

Suppose that

$$|\theta - \phi_0| < \frac{\alpha}{2}.$$

If $\phi \in [\phi_0 + \alpha, \phi_0 - \alpha + 2\pi]$ then

$$|\theta - \phi| > \frac{\alpha}{2}.$$

and so the denominator of the Poisson kernel

$$\frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2}$$

is bounded away from zero, say by m.

We have

$$\begin{aligned}
|I_2| &\leq \frac{1}{2\pi}\int_{\phi_0+\alpha}^{\phi_0-\alpha+2\pi} P_r(\phi - \theta)|h(e^{i\phi}) - h(e^{i\phi_0})|\,\mathrm{d}\phi \\
&\leq \frac{(1 - r^2)M}{2\pi m}2\pi \\
&\leq \frac{(1 - r)(1 + r)M}{m} \\
&\leq \frac{2M}{m}\delta \\
&= \frac{\epsilon}{2},
\end{aligned}$$

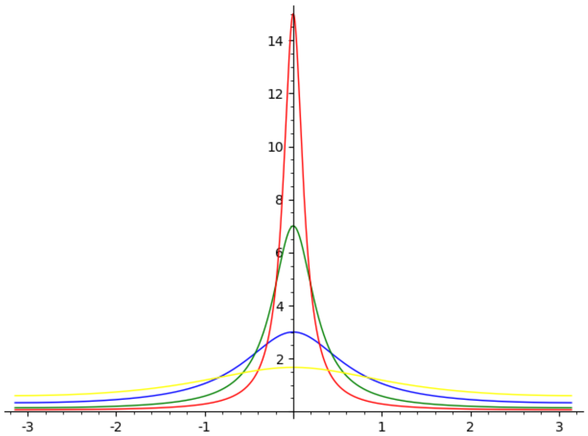
provided

$$|1 - r| < \delta \qquad \text{where} \qquad \delta = \frac{m\epsilon}{4M}.$$

Putting all of this together we get

$$\begin{aligned}
|u(r, \theta) - h(e^{i\phi_0})| &= |I_1 + I_2| \\
&\leq |I_1| + |I_2| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Note that the Poisson has a spike close to $e^{i\phi_0}$, meaning r is close to 1 and θ is close to ϕ_0, and most of the integral is concentrated there. For example, consider the graph of the Poisson integral when $\theta = 0$, so the peak is at 0. The axes go from $-\pi$ to π. The yellow curve is $r = 1/4$, the blue curve is $r = 1/2$, the green curve is $r = 3/4$ and the red curve is $r = 7/8$:



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