2. Contour Integration

Contour integration is a method to compute definite integrals by integrating a function with isolated singularities on a region around the boundary of the region and applying the residue theorem.

Typically the contour we integrate over gets larger and we have to identify various terms. There is an art to choosing the correct contour.

Example 2.1. *Calculate*

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2 + a^2} \qquad \textit{where} \qquad a > 0.$$

We integrate over the contour

$$\gamma = \gamma_1 + \gamma_2,$$

where γ_1 traverses the interval $[-R, R]$ and γ_2 traverses the semicircle in the upper half plane going from R to $-R$. γ is the boundary of the open disk centred at 0 of radius R intersected with the upper half plane $\mathbb{H}$. Call this region U.

The integrand is the function

$$f(z) = \frac{1}{z^2 + a^2}.$$

This has isolated singularities at $\pm ia$ and is otherwise holomorphic on the whole complex plane. If $R > a$ then ia belongs to U but not $-ia$.

We calculate the residue at ia. As $f(z)$ has a simple pole at ia we have

$$\begin{aligned}
\operatorname{Res}_{ia} f(z) &= \lim_{z \to ia} \frac{(z - ia)}{z^2 + a^2} \\
&= \lim_{z \to ia} \frac{1}{2z} \\
&= \lim_{z \to ia} \frac{1}{2ia} \\
&= -\frac{i}{2a}.
\end{aligned}$$

The residue theorem then implies that

$$\begin{aligned}
\int_{\gamma} \frac{\mathrm{d}z}{z^2 + a^2} &= 2\pi i \operatorname{Res}_{ia} f(z) \\
&= 2\pi i - \frac{i}{2a} \\
&= \frac{\pi}{a}.
\end{aligned}$$

On the other hand

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}z}{z^2+a^2} &= \int_{\gamma_1+\gamma_2} \frac{\mathrm{d}z}{z^2+a^2} \\ &= \int_{\gamma_1} \frac{\mathrm{d}z}{z^2+a^2} + \int_{\gamma_2} \frac{\mathrm{d}z}{z^2+a^2}.\end{aligned}$$

We attack both line integrals separately. If we paramaterise γ_1 in the obvious way then we get

$$\int_{\gamma_1} \frac{\mathrm{d}z}{z^2+a^2} = \int_{-R}^{R} \frac{\mathrm{d}x}{x^2+a^2}.$$

Note that the original integral is an improper integral which converges. So if we let R go to infinity then we get the integral we want.

This hides a subtle point. To check the improper integral converges we have to check that the double limit,

$$\lim_{l\to-\infty, u\to\infty} \int_l^u \frac{\mathrm{d}x}{(x^2+a^2)^2},$$

converges. To check this, the easiest thing is to split the integral into two pieces

$$\int_l^u \frac{\mathrm{d}x}{(x^2+a^2)^2} = \int_l^0 \frac{\mathrm{d}x}{(x^2+a^2)^2} + \int_0^u \frac{\mathrm{d}x}{(x^2+a^2)^2},$$

and check that both terms on the RHS converge as let l go to $-\infty$ and u go to ∞. To check these limits converge, for example to check the second limit converges, we simply compare with

$$\int_1^\infty \frac{\mathrm{d}x}{x^2}$$

which we can check converges diectly, since when we integrate $1/x^2$

$$\begin{aligned}\int_1^\infty \frac{\mathrm{d}x}{x^2} &= \lim_{l\to\infty} \int_1^l \frac{\mathrm{d}x}{x^2} \\ &= \lim_{l\to\infty} \left[-\frac{1}{x}\right]_1^l \\ &= \lim_{l\to\infty} -\frac{1}{l} + 1 \\ &= 1.\end{aligned}$$

Once we *know* the improper integral converges then we can let $l = u = R$ approach infinity simultaneously and we will get the integral we are trying to compute.

For the integral over γ_2 we try to bound the absolute value of the integral from above. The length L of γ_2 is πR. The maximum value M of the absolute value of $f(z)$ is

$$\begin{aligned} |f(z)| &= \left|\frac{1}{z^2+a^2}\right| \\ &= \frac{1}{|z^2+a^2|} \\ &\leq \frac{1}{R^2-a^2}. \end{aligned}$$

It follows that

$$\begin{aligned} \left|\int_{\gamma_2} \frac{\mathrm{d}z}{z^2+a^2}\right| &\leq LM \\ &\leq \frac{\pi R}{R^2-a^2}. \end{aligned}$$

The key point is that this rational fraction is bottom heavy, so that as R goes to infinity the rational function goes to zero. It follows that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+a^2} = \frac{\pi}{a}.$$

One can check this using a trig substitution.

Example 2.2. *Calculate*

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^2} \qquad \textit{where} \qquad a > 0.$$

We integrate over the same contour as before. The integrand is now

$$f(z) = \frac{1}{(z^2+a^2)^2}.$$

This has isolated singularities at $\pm ia$ and is otherwise holomorphic on the whole complex plane. If $R > a$ then ia belongs to U but not $-ia$.

As before we get two integrals. If we paramaterise γ_1 in the obvious way then we get

$$\int_{\gamma_1} \frac{\mathrm{d}z}{(z^2+a^2)^2} = \int_{-R}^{R} \frac{\mathrm{d}x}{(x^2+a^2)^2}.$$

Note that the original improper integral converges. So if we let R go to infinity then we get the integral we want.

We calculate the residue at ia. As $f(z)$ has a double pole at ia we have

$$\begin{aligned}
\operatorname{Res}_{ia} f(z) &= \lim_{z\to ia} \frac{\mathrm{d}}{\mathrm{d}z} \frac{(z-ia)^2}{(z^2+a^2)^2} \\
&= \lim_{z\to ia} \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{(z+ia)^2} \\
&= \lim_{z\to ia} -\frac{2}{(z+ia)^3} \\
&= -\frac{2}{(2ia)^3} \\
&= -\frac{i}{4a^3}.
\end{aligned}$$

The residue theorem then implies that

$$\begin{aligned}
\int_\gamma \frac{\mathrm{d}z}{(z^2+a^2)^2} &= 2\pi i \operatorname{Res}_{ia} f(z) \\
&= 2\pi i - \frac{i}{4a^3} \\
&= \frac{\pi}{2a^3}.
\end{aligned}$$

For the integral over γ_2 we try to bound the absolute value of the integral from above. The length L of γ_2 is still πR. The maximum value M of the absolute value of $f(z)$ is

$$\begin{aligned}
|f(z)| &= \left| \frac{1}{(z^2+a^2)^2} \right| \\
&= \frac{1}{|z^2+a^2|^2} \\
&\leq \frac{1}{(R^2-a^2)^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left| \int_{\gamma_2} \frac{\mathrm{d}z}{z^2+a^2} \right| &\leq LM \\
&\leq \frac{\pi R}{(R^2-a^2)^2}.
\end{aligned}$$

As before the rational fraction is bottom heavy, so that as R goes to infinity the rational function goes to zero. It follows that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+a^2} = \frac{\pi}{2a^3}.$$

Here is a way to check this calculation. Suppose that we differentiate both sides of
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+a^2} = \frac{\pi}{a},$$
with respect to a. We get
$$\begin{aligned}
-\frac{\pi}{a^2} &= \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{\pi}{a}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}a}\left(\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+a^2}\right) \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial a}\left(\frac{1}{x^2+a^2}\right)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} -\frac{2a}{(x^2+a^2)^2}\mathrm{d}x \\
&= -2a\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^2} \\
&= -2aI,
\end{aligned}$$
to get from the second line to the third line, we differentiated under the integral sign.

Solving for I, it follows that
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+a^2)^2} = \frac{\pi}{2a^3},$$
as before.

Example 2.3. *Calculate*
$$\int_{-\infty}^{\infty} \frac{x^2\mathrm{d}x}{(x^2+1)^2}.$$

We integrate over the standard contour. The integrand is now
$$f(z) = \frac{z^2}{(z^2+1)^2}.$$

This has isolated singularities at $\pm i$ and is otherwise holomorphic on the whole complex plane. If $R > 1$ then i belongs to U but not $-i$.

As before we get two integrals. If we paramaterise γ_1 in the obvious way then we get
$$\int_{\gamma_1} \frac{z^2\mathrm{d}z}{(z^2+1)^2} = \int_{-R}^{R} \frac{x^2\mathrm{d}x}{(x^2+1)^2}.$$
Note that the original improper integral converges. So if we let R go to infinity then we get the integral we want.

We calculate the residue at i. As $f(z)$ has a double pole at i we have

$$\begin{aligned}
\operatorname{Res}_i f(z) &= \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z} \frac{(z-i)^2 z^2}{(z^2+1)^2} \\
&= \lim_{z\to i} \frac{\mathrm{d}}{\mathrm{d}z} \frac{z^2}{(z+i)^2} \\
&= \lim_{z\to i} \frac{2z(z+i)^2 - 2z^2(z+i)}{(z+i)^4} \\
&= \lim_{z\to i} 2z \frac{(z+i)-z}{(z+i)^3} \\
&= \lim_{z\to i} 2z \frac{i}{(z+i)^3} \\
&= \frac{2i^2}{(2i)^3} \\
&= -\frac{i}{4}.
\end{aligned}$$

The residue theorem then implies that

$$\begin{aligned}
\int_\gamma \frac{z^2 \mathrm{d}z}{(z^2+1)^2} &= 2\pi i \operatorname{Res}_i f(z) \\
&= 2\pi i - \frac{i}{4} \\
&= \frac{\pi}{2}.
\end{aligned}$$

For the integral over γ_2 we try to bound the absolute value of the integral from above. The length L of γ_2 is still πR. The maximum value M of the absolute value of $f(z)$ is

$$\begin{aligned}
|f(z)| &= \left| \frac{z^2}{(z^2+1)^2} \right| \\
&= \frac{|z|^2}{|z^2+1|^2} \\
&\le \frac{R^2}{(R^2-1)^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left| \int_{\gamma_2} \frac{\mathrm{d}z}{z^2+a^2} \right| &\le LM \\
&\le \frac{\pi R^3}{(R^2-a^2)^2}.
\end{aligned}$$

As before the rational fraction is bottom heavy, so that as R goes to infinity the rational function goes to zero. It follows that

$$\int_{-\infty}^{\infty} \frac{x^2 \mathrm{d}x}{(x^2+1)^2} = \frac{\pi}{2}.$$

We can check this calculation using (2.1) and (2.2) with $a = 1$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2 \mathrm{d}x}{(x^2+1)^2} &= \int_{-\infty}^{\infty} \frac{(x^2+1) \mathrm{d}x}{(x^2+1)^2} - \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)} - \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^2} \\
&= \pi - \frac{\pi}{2} \\
&= \frac{\pi}{2}.
\end{aligned}$$