19. Dirichlet problem and the Poisson kernel

Suppose we are given a region U. Suppose that u is a harmonic function on U that extends to a continuous function on the boundary.

The **Dirichlet problem** is to reverse this process, starting with the continuous function on the boundary to exend this continuous function to a harmonic function on the whole region. This is one of the most well-studied boundary value problems in mathematics.

It seems worth starting with the following observation.

Theorem 19.1. *Let U be a bounded region and let h be a continuous function on the boundary of U,*

$$h \colon \partial U \longrightarrow \mathbb{R}.$$

Then there is at most one harmonic function u on U which extends to a continuous function on $U \cup \partial U$ such that the restriction of u to the boundary ∂U is equal to the function h on the boundary.

Proof. We first show that if h is identically zero then u is identically zero.

By the maximum principle applied to u we have that the maximum of u on $U \cup \partial U$ is achieved on the boundary. As u is zero on the boundary it follows that $u \leq 0$ on U.

By the minimum principle applied to u (or what comes to the same thing, the maximum principle applied to $-u$) the minimum of u on $U \cup \partial U$ is achieved on the boundary. As u is zero on the boundary it follows that $u \geq 0$ on U.

As $0 \leq u \leq 0$ it follows that u is identically zero.

Now we turn to the general case. Let u_1 and u_2 be two harmonic functions on U which both extend to continuous functions on $U \cup \partial U$ and which both restrict to h on the boundary.

Then $u = u_1 - u_2$ is a harmonic function on U which extends to a continuous function on $U \cup \partial U$ and which restricts to $h - h = 0$ on the boundary. By what we already proved u is identically zero. But then $u_1 = u_2$. □

It is a general phenomena in mathematics that problems with unique solutions behave much better than problems whose solutions are not unique. We start with the simplest region we know, the unit disk.

We start with Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)\,\mathrm{d}w}{w - z},$$

where $f(z)$ is a holomorphic function on the closed unit disk. We write z in polar coordinates,

$$z = re^{i\theta}.$$

Let

$$\zeta = r^{-1}e^{i\theta}.$$

ζ is called the **inverse point** of z, in the circle $|w| = 1$. It lies on the same line through the centre of the circle as z but it lies on a different side of the circle and we have

$$|z| \cdot |\zeta| = 1.$$

Since ζ is outside the circle, we have

$$\int_{|w|=1} \frac{f(w)\,\mathrm{d}w}{w - \zeta} = 0.$$

Combining this with the previous expression we get

$$f(z) = \frac{1}{2\pi i} \int_{|w|=1} \left(\frac{1}{w-z} - \frac{1}{w-\zeta} \right) f(w)\,\mathrm{d}w.$$

If use the standard parametrisation $w = e^{i\phi}$ then $\mathrm{d}w = iw\,\mathrm{d}\phi$ and we get

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{w}{w-z} - \frac{w}{w-\zeta} \right) f(w)\,\mathrm{d}\phi,$$

where we retain the symbol w for notational convenience. Note that

$$\zeta = \frac{1}{\bar{z}} \qquad \text{and} \qquad \bar{w} = \frac{1}{w},$$

since ζ is the inverse point and w is its own inverse point. It follows that the expression in brackets is

$$\begin{aligned}
\frac{w}{w-z} - \frac{w}{w-\zeta} &= \frac{w}{w-z} - \frac{w}{w-1/\bar{z}} \\
&= \frac{w}{w-z} - \frac{\bar{z}}{\bar{z}-\bar{w}} \\
&= \frac{w}{w-z} + \frac{\bar{z}}{\bar{w}-\bar{z}} \\
&= \frac{1-r^2}{|w-z|^2}.
\end{aligned}$$

It follows that we can rewrite Cauchy's integral formula as

$$f(z) = \frac{1-r^2}{2\pi} \int_0^{2\pi} \frac{f(e^{i\phi})}{|w-z|^2}\,\mathrm{d}\phi.$$

A priori this is only valid for $r \neq 0$, that is $z \neq 0$, since the inverse point of 0 is ∞, but in fact it does gives the correct formula even if $r = 0$.

Now the expression

$$|w - z|$$

is the distance between the point z and the point w on the unit circle. We can use the law of cosines to get

$$|w - z|^2 = 1 - 2r\cos(\phi - \theta) + r^2.$$

Finally, if $u(r, \theta)$ is the real part of f then we get the **Poisson integral formula**

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)u(1, \phi)}{1 - 2r\cos(\phi - \theta) + r^2} \, \mathrm{d}\phi.$$

Note that this integral expresses $u(r, \theta)$ in terms of the values $u(1, \phi)$ of u on the boundary of the circle. The factor

$$P_r(\phi - \theta) = \frac{1 - r^2}{1 - 2r\cos(\phi - \theta) + r^2}$$

is called the **Poisson kernel**. Of course we also have

$$P_r(\phi - \theta) = \frac{1 - r^2}{|w - z|^2}.$$

The Poisson kernel enjoys some basic properties.

Lemma 19.2.
(a) $P_r(\psi) > 0$.
(b)

$$P_r(\phi - \theta) = \mathrm{Re}\left(\frac{w + z}{w - z}\right).$$

(c) If we fix w then $P_r(\phi - \theta)$ is a harmonic function on the unit disk.
(d) $P_r(\psi)$ is an even, periodic function of ψ with period 2π.
(e) $P_0(\psi) = 1$.
(f)

$$\frac{1}{2\pi} \int_0^{2\pi} P_r(\phi - \theta) \, \mathrm{d}\phi = 1.$$

Proof. (d) and (e) follow from

$$P_r(\psi) = \frac{1 - r^2}{1 - 2r\cos\psi + r^2}$$

For (a) note that the numerator is positive. On the other hand,

$$\begin{aligned}
1 - 2r\cos\psi + r^2 &= 1 - 2r + r^2 + 2r - 2r\cos\psi \\
&= (r-1)^2 + 2r(1-\cos\psi),
\end{aligned}$$

so that the denominator is positive as well.

(b) We already showed that

$$P_r(\phi - \theta) = \frac{w}{w-z} + \frac{\bar{z}}{\bar{w}-\bar{z}}.$$

It follows that the RHS is real.

Note that

$$\operatorname{Re}\left(\frac{\bar{z}}{\bar{w}-\bar{z}}\right) = \operatorname{Re}\left(\frac{z}{w-z}\right).$$

Thus

$$\begin{aligned}
P_r(\phi-\theta) &= \operatorname{Re}\left(\frac{w}{w-z} + \frac{\bar{z}}{\bar{w}-\bar{z}}\right) \\
&= \operatorname{Re}\left(\frac{w}{w-z} + \frac{z}{w-z}\right) \\
&= \operatorname{Re}\left(\frac{w+z}{w-z}\right).
\end{aligned}$$

(c) $P_r(\phi - \theta)$ is harmonic as it is the real part of a holomorphic function on the unit circle.

Note that the function $u(r,\theta) = 1$ is a harmonic extension to the function $u(1,\theta) = 1$ on the circle $|z| = 1$. Thus

$$\begin{aligned}
1 &= u(r,\theta) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)u(1,\phi)}{1-2r\cos(\phi-\theta)+r^2}\,\mathrm{d}\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\,\mathrm{d}\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} P_r(\phi-\theta)\,\mathrm{d}\phi = 1.
\end{aligned}$$

□