18. Meromorphic versus Rational functions

One obvious way to give a meromorphic function on $\mathbb{C}$ is to write down a rational function, the quotient of two polynomials. It is natural to wonder to what extent these give all meromorphic functions on $\mathbb{C}$:

Theorem 18.1. *Let f be a meromorphic function on the whole of $\mathbb{C}$.*

Then f is a rational function if and only if it has at worse a pole at infinity.

Here we allow the possibility that f is holomorphic at infinity.

It is clear that we need the condition at infinity. For example the exponential function

$$z \longrightarrow e^z$$

is an entire function but it is is not a rational function. In fact it has an essential singularity at infinity.

We will need:

Definition 18.2. *Let f be a holomorphic function with an isolated singularity at a.*

*The **principal part** of f at a is the negative part of the Laurent series expansion of f at a.*

Proof of (18.1). One direction is clear. If

$$f(z) = \frac{p(z)}{q(z)}$$

is a rational function then

$$f\left(\frac{1}{z}\right) = \frac{p(1/z)}{q(1/z)}.$$

The RHS expands to a rational function in z. In particular $f(z)$ has a pole at infinity.

Now suppose that $f(z)$ is a meromorphic function with a pole at infinity. First observe that $f(z)$ has only finitely many singularities. Isolated singularities cannot accumulate anywhere. If there were infinitely many singularities their modulus would have to go to infinity. But as we have a pole at infinity this cannot happen either.

Let $w = 1/z$. By assumption

$$f(w) = \frac{a_{-n}}{w^n} + \frac{a_{-n+1}}{w^{n-1}} + \dots + a_0 + a_1 w + \dots.$$

The principal part at $w = 0$ is then

$$\frac{a_{-n}}{w^n} + \frac{a_{-n+1}}{w^{n-1}} + \dots + \frac{a_{-2}}{w^2} + \frac{a_{-1}}{w}.$$

If we subsitute back in $z = 1/w$ then we get a polynomial in z, $p_\infty(z)$.

Let $a_1, a_2, \dots, a_m$ be the finitely many other singular points. Let $p_{a_i}(z)$ be the principal part at a_i. Then $p_{a_i}(z)$ is a polynomial in $1/(z - a_i)$.

Consider

$$g(z) = f(z) - p_\infty(z) - p_{a_1}(z) - p_{a_2}(z) - \dots - p_{a_m}(z).$$

Note that

$$p_\infty(z) + p_{a_1}(z) + p_{a_2}(z) + \dots + p_{a_m}(z).$$

is a rational function in z, since it is a sum of rational functions. In particular it is a meromorphic function so that $g(z)$ is a meromorphic function.

The only possible poles of $g(z)$ are at $a_1, a_2, \dots, a_m$. But by construction the principal part of $g(z)$ at a_i is zero. Therefore $g(z)$ is holomorphic at a_i, so that it is an entire function.

On the other hand, $g(z)$ does not have a pole at infinity either. Therefore $g(z)$ is bounded as z approaches infinity. But then Liouville's theorem implies that $g(z)$ is a constant c. It follows that

$$f(z) = p_\infty(z) + p_{a_1}(z) + p_{a_2}(z) + \dots + p_{a_m}(z) + c,$$

is a rational function. □

It seems worth pointing out the decomposition above in p_∞ and p_{a_1}, p_{a_2}, ..., is essentially the decomposition of a rational function into partial fractions.

Example 18.3. *Obtain the partial fraction decomposition of*

$$\frac{z^3}{z^2+1}.$$

The first thing to do is figure out the term $p_\infty(z)$. We could subsitute $w = 1/z$, find the Laurent series expansion and then isolate the principal part. Much easier is to use the division algorithm. We try to divide $z^2 + 1$ into z^3. We get z with a remainder of

$$z^3 - z(z^2+1) = -z \qquad \text{so that} \qquad \frac{z^3}{z^2+1} = z - \frac{z}{z^2+1}.$$

Thus $p_\infty(z) = z$. Now

$$-\frac{z}{z^2+1}$$

has poles at $\pm i$. We have simple poles there and so

$$\begin{aligned} -\frac{z}{z^2+1} &= p_i(z) + p_{-i}(z) \\ &= \frac{\alpha}{z-i} + \frac{\beta}{z+i}. \end{aligned}$$

Now α is the residue at i and β is the residue at $-i$. We have

$$\begin{aligned} \alpha &= \lim_{z \to i} -\frac{z}{2z} \\ &= -\frac{1}{2}. \end{aligned}$$

It is clear the answer won't change at $-i$. Thus

$$\frac{z^3}{z^2+1} = z - \frac{1}{2}\frac{1}{z-i} - \frac{1}{2}\frac{1}{z+i}.$$

Of course we could do the last step using one of the usual methods.