17. Inverse Laplace Transforms

This is one more topic to do with contour integration. In this discussion we will ignore all technicalities, such as types of convergence and make some assumptions which are not always justified, even if the conclusion holds.

We recall the definition of:

Definition 17.1. *Let*

$$f\colon [0,\infty) \longrightarrow \mathbb{C}$$

be a continuous function.

The ***Laplace transform*** *of $f(t)$ is the function given by the integral*

$$F(s) = \int_0^{\infty} e^{-st} f(t)\,\mathrm{d}t.$$

Here s is a complex variable and $F(s)$ is a holomorphic function with finitely many singularities. The Laplace transform is similar to the Fourier transform and it can be used to solve partial differential equations.

Given $F(s)$ we would like to be able to recover $f(t)$, that is, we would like to compute the inverse Laplace transform. We give a recipe to do this but we will not justify the recipe.

Pick real numbers γ and R. Consider the vertical line segment L_R from $s = \gamma - iR$ to $s = \gamma + iR$. We choose γ so large that every singular point of $F(s)$ lies to the left of γ.

We define a function

$$f\colon [0,\infty) \longrightarrow \mathbb{C}$$

by the formula

$$f(t) = \frac{1}{2\pi i}\int_{L_R} e^{st} F(s)\,\mathrm{d}s.$$

If we parametrise L_R in the usual way then we are trying to compute the Cauchy principal value of

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\,\mathrm{d}s.$$

This integral is sometimes referred to as a **Bromwich integral**.

We won't justify that this is the correct formula to recover f from its Laplace transform. We will show how to compute the Bromwich integral, giving a way to compute f given F.

Consider the semicircle C_R going from $\gamma + iR$ to $\gamma - iR$ which is to the left of the line segment. Let U be the bounded open set whose boundary is $L_R + C_R$. We pick R so large that it captures all isolated singularities of $F(s)$, $a_1, a_2, \ldots, a_n$. If we apply the residue theorem then we get

$$\int_{L_R} e^{st} F(s)\,\mathrm{d}s + \int_{C_R} e^{st} F(s)\,\mathrm{d}s = 2\pi i \sum_{i=1}^{n} \mathrm{Res}_{a_i}\, e^{st} F(s)$$

If we are in luck then the second term on the LHS goes to zero, as R goes to infinity and the first term approaches the Bromwich integral.

Example 17.2. *Compute the inverse Laplace transform of*

$$F(s) = \frac{s}{s^2+4}.$$

$F(s)$ is a rational function with simple poles at $\pm 2i$. We take $\gamma = 1$.

We compute the residues of $e^{st}F(s)$ at $\pm 2i$. As the residues are simple poles we have

$$\begin{aligned} \mathrm{Res}_{2i} \frac{se^{st}}{s^2+4} &= \lim_{s \to 2i} \frac{se^{st}}{s+2i} \\ &= \frac{2ie^{2it}}{4i} \\ &= \frac{e^{2it}}{2} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Res}_{-2i} \frac{se^{st}}{s^2+4} &= \lim_{s \to -2i} \frac{se^{st}}{s-2i} \\ &= \frac{-2ie^{-2it}}{-4i} \\ &= \frac{e^{-2it}}{2} \end{aligned}$$

We assume that the integral around the semicircle goes to zero, as R goes to infinity. In this case the residue theorem implies that

$$\begin{aligned} f(t) &= \mathrm{Res}_{2i} \frac{se^{st}}{s^2+4} + \mathrm{Res}_{-2i} \frac{se^{st}}{s^2+4} \\ &= \frac{e^{2it}}{2} + \frac{e^{-2it}}{2} \\ &= \cos 2t. \end{aligned}$$

We check this by computing the Laplace transform of $\cos 2t$. It is easier to compute the Laplace transform of e^{2it} and e^{-2it} and use linearity. We assume that $\mathrm{Re}(s) > 0$ to ensure convergence. The

Laplace transform of e^{2it} is

$$\begin{aligned}\int_0^\infty e^{-st}e^{2it}\,\mathrm{d}t &= \int_0^\infty e^{(2i-s)t}\,\mathrm{d}t\\ &= \frac{1}{2i-s}\left[e^{(2i-s)t}\right]_0^\infty\\ &= -\frac{1}{2i-s}.\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^\infty e^{-st}e^{-2it}\,\mathrm{d}t &= \int_0^\infty e^{(-2i-s)t}\,\mathrm{d}t\\ &= -\frac{1}{2i+s}\left[e^{(2i-s)t}\right]_0^\infty\\ &= \frac{1}{2i+s}.\end{aligned}$$

If we add the results together and divide by 2 we get the Laplace transform of $\cos 2t$:

$$\begin{aligned}\frac{1}{2}\left(\frac{1}{2i+s}-\frac{1}{2i-s}\right) &= \frac{1}{2}\left(\frac{2i-s-2i-s}{(2i+s)(2i-s)}\right)\\ &= \frac{s}{s^2+4},\end{aligned}$$

as expected.