16. Conformal maps of the unit disc

Recall that a map from one region to another is called **conformal** if it is a bijection and it preserves angles. This is equivalent to saying that the map is a bijection and it is holomorphic with nowhere zero derivative.

Definition 16.1. *Let $f\colon U \longrightarrow V$ be a holomorphic map between two regions.*

*We say that f is **biholomorphic** if there is a holomorphic map $g\colon V \longrightarrow U$ such that $g \circ f\colon U \longrightarrow U$ and $f \circ g\colon V \longrightarrow V$ are both the identity.*

As usual, we will sometimes refer to g as the inverse map of f.

Lemma 16.2. *A map between two regions is a conformal equivalence if and only if it is biholomorphic.*

Proof. Suppose that $f\colon U \longrightarrow V$ is a conformal equivalence. Then f is holomorphic and its derivative is nowhere zero. As f is a bijection there is an inverse map $g\colon V \longrightarrow U$. As f preserves angles then so does g. Thus g is holomorphic. It follows that f is biholomorphic.

Now suppose that f is biholomorphic. Then f is holomorphic and there is a holomorphic map $g\colon V \longrightarrow U$ such that $f \circ g$ and $g \circ f$ are the identity. But then f is certainly a bijection.

We have

$$(g \circ f)(z) = z.$$

Thus the chain rule implies that

$$g'(f(z))f'(z) = 1.$$

In particular $f'(z)$ is nowhere zero. It follows that f is a conformal equivalence. $\square$

Given any object in mathematics, it is always interesting to write down all of its symmetries. Let us start with the unit disk. What are the biholomorphic maps from the unit disk to itself?

One possibility is simply to rotate through an angle. The map

$$z \longrightarrow e^{i\varphi} z$$

is a biholomorphic map of the unit disk to itself. The inverse map is

$$z \longrightarrow e^{-i\varphi} z.$$

Lemma 16.3. *If $g\colon \Delta \longrightarrow \Delta$ is a biholomorphic map such that $g(0) = 0$ then $g(z) = e^{i\varphi} z$ is a rotation.*

Proof. We first apply Schwarz's Lemma to $g(z)$. $g(z)$ is a holomorphic map on the unit disk, $|g(z)| < 1$ and $g(0) = 0$. It follows that

$$|g(z)| \leq |z|$$

with equality if and only if g is a rotation.

Now consider the inverse map $h\colon \Delta \longrightarrow \Delta$. $h(w)$ is a holomorphic map on the unit disk, $|h(w)| < 1$ and $h(0) = 0$. It follows that

$$|h(w)| \leq |w|$$

with equality if and only if h is a rotation.

Then

$$\begin{aligned} |z| &= |h(g(z))| \\ &= |h(w)| \\ &\leq |w| \\ &= |g(z)| \\ &\leq |z|. \end{aligned}$$

As we have a sring of inequalities and the extremes are equal, it follows that all of the inequalities above are equalities. But then $g(z)$ is a rotation. $\square$

Can we find interesting biholomorphic maps of the unit disk to itself which are not rotations? By (16.3) we are looking for maps that don't send the origin to the origin. One thing to try is a Möbius transformation. Consider

$$M(z) = \frac{z - a}{1 - \bar{a}z} \qquad \text{where} \qquad a \in \Delta.$$

This is a Möbius transformation. It is meromorphic and holomorphic except at $1/\bar{a} \notin \Delta$. Möbius transformations permute lines and circles. Suppose that we take a point $e^{i\theta}$ on the unit circle. Then

$$\begin{aligned} |e^{i\theta} - a| &= |e^{-i\theta} - \bar{a}| \\ &= |e^{-i\theta}| \cdot |1 - \bar{a}e^{i\theta}| \\ &= |1 - \bar{a}e^{i\theta}|. \end{aligned}$$

It follows that

$$\begin{aligned} |M(e^{i\theta})| &= \left| \frac{e^{i\theta} - a}{1 - \bar{a}e^{i\theta}} \right| \\ &= \frac{|e^{i\theta} - a|}{|1 - \bar{a}e^{i\theta}|} \\ &= 1, \end{aligned}$$

so that M takes the unit circle to the unit circle. As $M(a) = 0$, M must send Δ to Δ. As Möbius transformations have inverses, it follows that M is a biholomorphic map of the unit disk to itself.

We will need a very simple calculation:

Lemma 16.4. *If*

$$M(z) = \frac{az+b}{cz+d}$$

is a Möbius transformation then

$$M'(z) = \frac{ad-bc}{(cz+d)^2}.$$

Proof. We have

$$\begin{aligned} M'(z) &= \frac{a(cz+d)-(az+b)c}{(cz+d)^2} \\ &= \frac{ad-bc}{(cz+d)^2}. \end{aligned}$$

□

Theorem 16.5. *The set of all biholomorphic maps of the unit disk to itself are precisely the Möbius transformations of the form*

$$f(z) = e^{i\varphi}\frac{z-a}{1-\bar{a}z},$$

where $a \in \Delta$ and $\varphi \in [0, 2\pi)$ are uniquely determined by $f(z)$.

Proof.

$$z \longrightarrow e^{i\varphi}\frac{z-a}{1-\bar{a}z}$$

is the composition of two biholomorphic maps of the unit disk to itself,

$$z \longrightarrow \frac{z-a}{1-\bar{a}z} \qquad \text{and} \qquad z \longrightarrow e^{i\varphi}z.$$

Therefore it is a biholomorphic map of the unit disk to itself.

Now suppose that

$$f\colon \Delta \longrightarrow \Delta$$

is a biholomorphic map. Suppose that $f(a) = 0$. Let N be the inverse of the map M. Then N is a Möbius transformation that takes 0 to a.

Then

$$g = f \circ N \colon \Delta \longrightarrow \Delta$$

sends 0 to 0. It follows that g is a rotation,

$$g(z) = e^{i\varphi}z.$$

for some φ. If we start with the equality

$$g = f \circ N$$

and precompose with M we get

$$g \circ M = f \circ N \circ M.$$

But $N \circ M$ is the identity so that

$$f = g \circ M.$$

It follows that f is the composition of M and g.

We now turn to uniqueness. Note that a is the inverse image of zero. The derivative of $f(z)$ is given by

$$f'(z) = e^{i\varphi}\frac{1-|a|^2}{(1-\bar{a}z)^2}.$$

Thus φ is determined as the argument of $f'(0)$. $\square$

Recall that given a Möbius transformation,

$$z \longrightarrow \frac{az+b}{cz+d},$$

we can associate a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

up to scalar matrices. Composition of Möbius transformations is the same as matrix multiplication and so the inverse of a Möbius transformations is given by computing the inverse matrix. For example, if

$$M(z) = \frac{z-a}{1-\bar{a}z},$$

where $a \in \Delta$ then the associated matrix is

$$\begin{pmatrix} 1 & -a \\ -\bar{a} & 1 \end{pmatrix}$$

The determinant of this matrix is $1-|a|^2 > 0$. The inverse matrix is

$$\frac{1}{1-|a|^2}\begin{pmatrix} 1 & a \\ \bar{a} & 1 \end{pmatrix}.$$

The corresponding Möbius transformation is

$$N(z) = \frac{z+a}{1+\bar{a}z}.$$

Note that the determinant plays no role, since the matrix is only defined up to scalars. It is clear that this has the same shape as M and one can check that N is the inverse of M.

Theorem 16.6 (Pick's Lemma)**.** *If $f\colon \Delta \longrightarrow \Delta$ is holomorphic then*

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2}$$

on the unit disk Δ.

Further, either $f(z)$ is biholomorphic, in which case equality holds everywhere, or the inequality is strict everywhere.

Proof. Pick $a \in \Delta$ and suppose that $b = f(a)$. Define

$$g(z) = \frac{z+a}{1+\bar{a}z} \qquad \text{and} \qquad h(w) = \frac{w-b}{1-\bar{b}w}.$$

Then g and h are biholomorphic maps $\Delta \longrightarrow \Delta$ and so the composition

$$h \circ f \circ g\colon \Delta \longrightarrow \Delta$$

is a holomorphic map such that

$$(h \circ f \circ g)(0) = 0.$$

Schwarz's Lemma implies

$$\begin{aligned} 1 &\geq |(h \circ f \circ g)'(0)| \\ &= |h'(b)f'(a)g'(0)| \\ &= |h'(b)| \cdot |f'(a)| \cdot |g'(0)|. \end{aligned}$$

Hence

$$|f'(a)| \leq \frac{1}{|h'(b)| \cdot |g'(0)|}.$$

But

$$g'(0) = 1 - |a|^2 \qquad \text{and} \qquad h'(b) = \frac{1}{1-|b|^2}.$$

This gives the first inequality.

If $f(z)$ is biholomorphic then so is the composition $h \circ f \circ g$ and the inequality in the first line above is an equality. Thus if $f(z)$ is biholomorphic then we have equality.

Now suppose that we have equality somewhere. Then the first line above must be an equality. But then

$$(h \circ f \circ g)(z) = \lambda z \qquad \text{where} \qquad |\lambda| = 1.$$

But then multiplying by the inverse of h on the left and the inverse of g on the right we see that f is biholomorphic. □