15. Holomorphic maps of the unit disc

Theorem 15.1 (Schwarz's Lemma)**.** *Let*

$$f\colon \Delta \longrightarrow \mathbb{C}$$

be a holomorphic map on the unit disk.

If $|f(z)| \leq 1$ *on the unit disk and* $f(0) = 0$ *then*

$$|f(z)| \leq |z| \qquad \text{on} \qquad \Delta,$$

with equality at some point $a \in \Delta$*,* $a \neq 0$*, if and only if* $f(z) = \lambda z$*, for some complex number* λ *with* $|\lambda| = 1$*.*

Proof. Consider the meromorphic function

$$g\colon \Delta \longrightarrow \mathbb{C} \qquad \text{given by} \qquad g(z) = \frac{f(z)}{z}.$$

It is clear that $g(z)$ is holomorphic, except possibly at 0.

As $f(z)$ is zero at 0, it follows that $f(z) = zh(z)$ where $h(z)$ is holomorphic at 0. If we compare equations then we get $g(z) = h(z)$, so that $g(z)$ is in fact holomorphic at 0.

Consider $g(z)$ on the closed disk of radius r, where $r \in (0, 1)$. $g(z)$ is holomorphic on this closed disk and so it is continuous on the boundary. It follows that it achieves its maximum on the boundary. But

$$\begin{aligned} |g(z)| &= \left|\frac{f(z)}{z}\right| \\ &= \frac{|f(z)|}{|z|} \\ &\leq \frac{1}{r}, \end{aligned}$$

on the boundary. It follows that

$$|g(z)| \leq \frac{1}{r},$$

on the closed disk of radius r. If we let r approach 1 from below then we get

$$|g(z)| \leq 1$$

on the unit disk. But then

$$|f(z)| \leq |z| \qquad \text{on} \qquad \Delta.$$

Now suppose we get equality, so that

$$|f(a)| = |a|.$$

In this case $|g(a)| = 1$ and so the strict maximum principle implies that g is constant. Suppose that $g(z) = \lambda$ for all z. Then

$$f(z) = \lambda z \qquad \text{where} \qquad |\lambda| = 1. \qquad \square$$

Here is a variant of Schwarz's Lemma with an arbitrary circle:

Lemma 15.2. *If $f(z)$ is holomorphic on the open disk*

$$U = \{ z \in \mathbb{C} \mid |z - a| < R \}$$

of radius R centred about a,

$$|f(z)| \leq M \qquad \text{and} \qquad f(a) = 0,$$

then

$$|f(z)| \leq \frac{M}{R}|z - a|$$

on U, with equality if and only if $f(z)$ is a multiple of $z - a$.

Proof. The idea is to reduce this to Schwarz's Lemma. Consider the map

$$\alpha \colon z \longrightarrow Rz + a.$$

This sends the unit disk to the disk U. Now consider the map

$$\beta \colon z \longrightarrow z/M.$$

The sends the disk of radius M centred at 0 to the unit disk. Both α and β are Möbius transformations.

It follows that if we put

$$g = \beta \circ f \circ \alpha \colon \Delta \longrightarrow \mathbb{C},$$

then g is a holomorphic function, as it is the composition of holomorphic functions,

$$\begin{aligned} g(0) &= \beta(f(\alpha(0))) \\ &= \beta(f(a)) \\ &= \beta(0) \\ &= 0, \end{aligned}$$

and

$$|g(w)| \leq 1.$$

We apply (15.1) to the function g. We conclude that

$$|g(w)| \leq |w|,$$

with equality if and only if $g(w)$ is a multiple of w. The inverse of β is

$$z \longrightarrow Mz$$

If we apply the inverse of β to both sides we get

$$|(f \circ \alpha)(w)| \leq |M||w|.$$

Pick $z \in U$. If we put

$$w = \frac{z-a}{R},$$

then $w \in \Delta$ and $\alpha(w) = z$. We have

$$\begin{aligned} |f(z)| &= |f(\alpha(w))| \\ &\leq M|w| \\ &= \frac{M}{R}|z-a|. \end{aligned}$$

Now suppose we have equality. Then $|g(w)| = |w|$, so that

$$g(w) = \lambda w$$

for some complex number of unit length. It follows that

$$\begin{aligned} f(z) &= Mg(w) \\ &= (M\lambda)w \\ &= \frac{M\lambda}{R}(z-a). \end{aligned}$$

□

There is also a version involving derivatives:

Theorem 15.3. *Let*

$$f \colon \Delta \longrightarrow \mathbb{C}$$

be a holomorphic map on the unit disk.

If $|f(z)| \leq 1$ on the unit disk and $f(0) = 0$ then

$$|f'(0)| \leq 1$$

with equality if and only if $f(z) = \lambda z$ for some complex number λ with $|\lambda| = 1$.

Proof. We have

$$\begin{aligned} |f'(0)| &= \left|\lim_{z \to 0} \frac{f(z)}{z}\right| \\ &= \lim_{z \to 0} \left|\frac{f(z)}{z}\right| \\ &= \lim_{z \to 0} \frac{|f(z)|}{|z|} \\ &\leq \lim_{z \to 0} 1 \\ &= 1. \end{aligned}$$

This establishes the inequality. Now suppose we have equality.

As in the proof of (15.1), we have

$$f(z) = zg(z)$$

where $g(z)$ is holomorphic. It follows that

$$f'(0) = g(0).$$

If

$$|f'(0)| = 1$$

then

$$|g(0)| = 1$$

and so $g(z)$ is constant, by the strict maximum principle. But then $f(z) = \lambda z$, where $\lambda = f'(0)$. $\square$